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THEORY OF THE IDEAL WINDMILL.

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The theory of the ideal propeller**, which was first established by FINSTERWALDER and afterwards completed by BENDEMANN and PRANDTL, is of fundamental importance in solving the problem of the action of the propeller. The same theory may also be applied to the ideal windmill, concerning which it has led to simple but explicit conclusions which may easily be utilized as the mature fruit of earlier work on the subject.

The following symbols*** have been adopted (see Fig. 1):

v , velocity of the airstream (in m/s)

p , static pressure in the free airstream, (in kg/m²)

ρ , density of the air (in kg s²/m⁴)

* This article (29th Report of the German Testing Laboratory for Aircraft) was written somewhat later than Dr-Phil. Max MUNK'S investigation of the subject, but was independent of same. It was published as a sequel to Dr. MUNK'S work in order to give additional prominence to the relation existing between propeller and windmill. Reference should also be made to "Utilization of the Wind by means of Windmills," published at about the same time by Dipl-Ing. Dr-Phil. Albert BETZ, of Göttingen in the "Zeitschrift für das gesamte Turbinenwesen," Vol. for 1920.

** }
*** } See next page.

Q, volume of air required per second in the free air-stream, (in m^3/S)

S, resistance of the windmill (P), (in kg)

L, theoretically available power (N), (in kg m/s),

P_{pr}, practically utilized power (in kg m/s).

In front of	:	in	:	behind	:
	:	the windmill	:		:
	:		:		:
F_0	:	F	:	F_1	:
	:		:		:
$w_0 = v$:	w (v_m)	:	$w_1 (v_2)$:
	:		:		:
$p_0 = p$:	p' p''	:	$p_1 = p$:
	:		:		:

slip stream section in m^2 .

velocity in m/s

static pressure in kg/m^2

$$\frac{w_1}{v} = \alpha = \frac{\text{velocity behind the windmill}}{\text{velocity of the wind}}$$

$$\frac{2 L}{\rho v^3 F} = \text{power coefficient } l, (k_n),$$

$$\frac{2 S}{\rho v^2 F} = \text{load coefficient } b, (o_3),$$

$$\frac{L}{S v} = \eta_{\text{th}} \text{ theoretical efficiency } (\eta)$$

$$\frac{L p_r}{S v} = \eta \text{ practical efficiency,}$$

$$\frac{L_{pr}}{L} = \frac{\eta}{\eta_{th}} = \xi \text{ coefficient of efficiency,}$$

$$\lambda = \frac{\text{wind velocity}}{\text{peripheral velocity}} = \text{coefficient of advance.}$$

** "Zeitschrift für Flugtechnik und Motorluftschiffahrt," 1910, p.177, and 1918, p.34. BENDEMANN'S "Luftschraubenuntersuchungen," No.1, (R. Oldenbourg, Munich,) 1911, p.10.

"Technische Berichte der Flugzeugmeisterei, Vol. II, p.53.

*** When MUNK'S symbols differ, they are inserted in brackets beside the other symbols.

By a method similar to that adopted by PRANDTL* for the propeller, any device may be considered as a windmill if it produce such change of pressure, on any surface F within a certain limited edge curve, that every particle of air penetrating the surface undergoes a reduction of pressure amounting to Δp constant. The limited edge curve is assumed, as in the case above named, to be a circle because more comprehensive calculations are thus obtained and also because the outlines of the windmill are entirely circular. The change of pressure produces a slipstream behind the windmill, within which the prevailing velocity is lower than beyond it.

If w be the velocity at any point of the surface, such velocity being generally variable from point to point, not perpendicular to the surface, Bernoulli's equation for the front part of the streamline is as follows:

$$p_0 + \frac{1}{2} \rho v^2 = p' + \frac{1}{2} \rho w^2 \dots \dots \dots (1)$$

and for the rear part of the streamline in question

$$p_0 + \frac{1}{2} \rho w_1^2 = p'' + \frac{1}{2} \rho w^2 \dots \dots \dots (2)$$

by subtracting from (1) and (2), we obtain:

$$\Delta p = p' - p'' = \frac{1}{2} \rho (v^2 - w_1^2) \dots \dots \dots (3)$$

The first application of the law of momentum to the section of change of pressure (dotted in Fig. 1) gives:

$$\begin{aligned} \rho F w^2 + p' F - (\rho F w^2 + p'' F) &= \Delta p \cdot F = \\ &= \frac{1}{2} \rho (v^2 - w_1^2) F = \frac{v^2}{2} \rho F (1 - \alpha^2) = S \dots \dots (4) \end{aligned}$$

* See "Technische Berichte," Vol. II, pp. 78.

By introducing the load coefficient:

$$S = \frac{b F v^2}{2} \dots \dots \dots (4a)$$

$$b = (1 - \alpha^2) \dots \dots \dots (5)$$

In Fig. 2, the sweep of the values of b is given in terms of α , which may vary between 0 and 1.

In this article the relation of the velocity behind the windmill to the velocity of the wind is represented in terms of α , as both velocities can be measured in the case of stationary windmills, whereas the thrust S can be determined with the aid of special auxiliary means only. It is not improbable that measuring devices might be contrived with a view to regulating such power.

In a second application of the law of momentum, on an extended control surface (dotted in Fig. 1) consisting of the streamline - which flows along the edge curve - and two end surfaces at the requisite distance from one another, it is noticeable that a similar pressure $p = p_0$ prevails in general beyond the control surface, and that the resultant of that pressure is consequently null. Variation in the momentum demands a thrust on the part of the machine inside the control section.

$$\rho F_0 v^2 + F_0 p_0 + (F_1 - F_0) p - F_1 p_1 - \rho F_1 w_1^2 = S. \dots (6)$$

As $p_0 = p_1 = p$ and as $Q = F_0 v = F w = F_1 w_1 = F_1 v \alpha$, then

$$\rho Q v - \rho Q w_1 = \rho Q (v - w_1) = \rho v Q (1 - \alpha) = S.$$

By combining the two values of S obtained by the aid of the law of momentum, we obtain:

$$Q = \frac{1}{2} Fv (1 + \alpha); w = \frac{1}{2} v (1 + \alpha) \dots \dots \dots (7)$$

and

$$F_1 = \frac{1}{2} F \frac{(1 + \alpha)}{\alpha} \dots \dots \dots (8)$$

also

$$\frac{2}{(1 + \alpha)} F_1 = F = \frac{2 \alpha}{(1 + \alpha)} F_1 \dots \dots \dots (9)$$

Applied to the small control section, the law of energy gives

$$\frac{w^2}{2} \rho Q + F p' w - \left\{ \frac{w^2}{2} \rho Q + F p'' w \right\} =$$

$$Q \Delta p = \frac{1}{4} \rho F v^3 (1 + \alpha) (1 - \alpha^2) = L \dots \dots \dots (10)$$

This result is based on the following considerations:

In order to obtain power from an air current, it is essential that the machine be held against the thrust S bearing upon it in the current, and for this purpose an expenditure S_w of power. From this expenditure, L is obtained. The amount

$$\frac{1}{2} \rho F_1 w_1 (v - w_1)^2 = \frac{1}{4} \rho F v^3 (1 + \alpha) (1 - \alpha)^2$$

remains in the slipstream. The three quantities are therefore connected by:

$$L = S_w - \frac{v^3}{4} \rho F (1 + \alpha) (1 - \alpha)^2 = \frac{1}{4} \rho F v^3 (1 + \alpha) (1 - \alpha^2) \dots \dots \dots (11)$$

By inserting the power coefficient, we obtain:

$$L = \frac{C_p \rho F v^3}{2} \dots \dots \dots (11a)$$

when

$$\ell = \frac{1}{2} (1 + \alpha) (1 - \alpha^2) \dots \dots \dots (12)$$

In Fig. 3, the sweep of ℓ is expressed in terms of α . ℓ attains its maximum when $\alpha = 1/3$, which may easily be proved by differentiating equation (12).

This gives:

$$\ell_{\max} = \frac{16}{27} = 0.5926.$$

and thus

$$\boxed{L_{\max} = \frac{8}{27} \rho F v^3} \dots \dots \dots (13)$$

Equation (13) states that the highest possible power of a windmill can be obtained only when the velocities in front of it and behind it are in the fixed ratio 3:1, and that the highest power apart from that condition, as well as the density of air, depends upon the third power of the velocity only.

The coefficient of efficiency theoretically attainable in a windmill is given by:

$$\eta_{th} = \frac{L}{S v} = \frac{\frac{v^3}{4} \rho F (1 + \alpha) (1 - \alpha^2)}{\frac{v^3}{2} \rho F (1 - \alpha^2)} = \frac{(1 + \alpha)}{2} \quad (14)$$

which is the equation of a straight line for different values of α (see Fig. 4). When $\alpha = 1/3$, L has its maximum value; but

$$\eta_{th} = \eta_{th \max} = 2/3 = 0.667 \dots \dots \dots (15)$$

This result is anything but satisfactory, as it unconditionally states that 1/3 of the wind-power cannot be util-

ized, even under ideal conditions, so that it wasted.

The greatest thrust that can be obtained from the windmill is attained when $\alpha = 0$. In that case:

$$b = b_{\max} = 1, \text{ and}$$

$$S_{\max} = \frac{1}{2} \rho v^2 F \dots \dots \dots (16)$$

In obtaining the highest power, i.e., $\alpha = \frac{1}{3}$, equation (5) shows that $b = 8/9$, and

$$S_{L_{\max}} = \frac{4}{9} \rho v^2 F \dots \dots \dots (17)$$

Values (16) and (17) differ from each other but slightly, as

$$\frac{S_{L_{\max}}}{S_{\max}} = \frac{8/9}{1} = 0.889.$$

Actual windmills cannot attain such high values, on account of losses due to friction and to eddies. Experiments made on good propellers, whereby efficiencies of $\zeta = 0.85$ to 0.90 have been attained, may justly be applied to windmills, and in that case similar or even still higher efficiencies - due to the low stream velocities - may be anticipated*. If $\zeta = 0.9$ be inserted, we obtain, as the practical coefficient of efficiency at maximum power:

$$\eta_{L_{\max}} = \zeta \eta_{th} L_{\max} = \frac{2 \cdot 0.9}{3} = 0.6 \dots (18)$$

and the actual maximum power practically attainable:

$$L_p r_{\max} = \zeta L_{\max} = \frac{8 \cdot 0.9}{37} \rho F v^3 \approx 0.26 \rho F v^3 \quad (19)$$

* MUNK obtains a coefficient of efficiency of $\zeta = \frac{0.88}{0.94} = \infty$ 0.94, - which is an extremely high value, - when $b(c_3) = \infty 0.25$. In the case of stationary windmills, an effort should be made to obtain efficiencies of about $1(k_n) = 0.59$.

Various conclusions are arrived at through the theory of the ideal windmill:

1. The maximum power attainable is frequently regarded* as being equal to the kinetic energy of the entire fluid stream traversing the surface of the windmill, when the following is inserted, as coefficient of efficiency:

$$\eta_{th} = \frac{L}{1/2 \rho v^3 \cdot F} = \frac{1}{2} (1 + \alpha) (1 - \alpha^2).$$

This ratio is in accordance with equation (12) for power coefficient C_p and is given in terms of α in Fig. 3.

Such interpretation may easily lead to erroneous suppositions if the expression "coefficient of efficiency" be retained as designation of the ratio of power obtained and power utilized, as the entire stream traverses the windmill surface F when $\alpha = 1$. In that case, however, $\eta_{th} = 1.00$ according to equation (14) and power is no longer withdrawn.

The denomination COEFFICIENT OF POWER provides, on the contrary, suitable expression for the estimation of a windmill.

2. When $w_1 = 0$ that is $\alpha = 0$, then

$$L = 1/4 \rho F v^3; \quad 2 F_0 = F = 0 F_1 \quad \text{and} \quad \eta_{th} = \frac{1}{2}$$

as shown in Fig. 5. The formula concords with the airflow of a fixed propeller as soon as the direction of the current is reversed. A "stationary" windmill of the kind would not be possible, on account of the airflow which is also

* See "Hütte," 22nd edition, Vol. II, p.2.

found beyond the limited inflow.

3. Assuming that $\alpha > 1$, the formulas enter the domain of the propeller. We then get:

$$S = - \frac{1}{2} v^2 \rho F (\alpha^2 - 1)$$

and

$$\dot{L} = - \frac{1}{4} \rho v^3 F (\alpha^2 - 1) (\alpha + 1)$$

and hence

$$\eta_{th} = \frac{S \cdot v}{\dot{L}} = \frac{2}{(\alpha + 1)}$$

The minus signs show that thrust is produced and power utilized. The relation between \mathcal{L} , b , and η_{th} curves for windmills and propellers is drawn in Fig. 6.

The formulas may easily be inserted in the customary formulas for the ideal windmill, in which the ratio of the airstream velocities is not usually calculated.

4. As it has been proved, in a general manner, that not more than 2/3 of the available wind-power can be utilized in a windmill stationed in a free airstream, the question as to whether some improvement might or might not be attained in this respect by controls, etc. meets with a negative reply. Should it be necessary, however, for practical reasons, to reduce the revolving parts of the machines as far as possible, experiments must be made with a view to determining whether the air available can be inducted into the windmill through a specially constructed nozzle at high velocity and with a small section. If this can be done, within certain limits, the resistance of the entire installation will be increased by the device, its losses being in-

creased and the total working coefficient diminished. It is, therefore, doubtful if the advantages offered by the reduced windmill are compensated by such disadvantages.

5. A windmill driven by a free airstream resembles a propeller more closely than a turbine. The propellers of simple type, with few blades, were most successful, and the same result may be anticipated for the wheels of the windmill.

As far back as 1907, LA COUR* recognized and discussed the fact that the working power of a windmill does not depend on the number of its blades and that an exaggerated number of blades, on the contrary, merely reduces its working power. He sums up his investigations of the test mill at Askov as follows: "The form which represents the highest working power is similar to that obtained by man through the experience of centuries, without any logical knowledge of the relative conditions." LA COUR rejects multi-bladed windmills and considers them as being favorable only to a certain extent on "starting" from a stationary position. This may be explained by the fact that the angle of attack of the stationary wheel with few blades is unfavorable. According to LA COUR, multi-bladed propellers must have larger pitch than those with few blades, so they consequently show less tendency to this drawback. It may here be mentioned that LA COUR first discovered the advantage of cambered

* "Ingenieuren," 1897, No. 10. Also "Windkraftmaschinen und ihre Anwendung zum Antrieb von Elektrizitätswerken," von Professor Paul LA COUR, Director of the test mill at Askov, near Vejen, translated into German by Dr. Joh. KAUFMANN, Leipzig, 1905, successor to M. HEINSIUS.

wings over plane surfaces, on his own initiative. He particularly recommends a section cambered at $1/4$ or $1/6$ of the depth of the leading edge, at 3 to 4% of the chord.

To what extent the power production of windmills may be improved by modern wing sections must be discovered by means of adequate and systematic tests, and the LA COUR conclusions with regard to the number of blades can only be confirmed in the same manner. The conclusions drawn by MUNK in respect of the number of blades do not confirm LA COUR'S supposition.

Ordinary propellers work with a coefficient of efficiency of $\ell = 0.3$ to 0.6 . The most favorable propeller efficiency coefficients being about $\ell = 0.59$, windmills of high power closely resemble propellers with high power charges.

6. The wind velocity v is considered, in this theory, as being uniform. The ordinary winds, that is, the wind close to the ground, are irregular within wide limits; they vary in mean value and are dependent upon locality and direction. The mean power of windmills in the wind is therefore proportional to the third power of the velocity; we must consequently take the means of the third power of the wind velocity in order to determine the mean power of the windmill, due to the wind. Fig. 7 shows the difference between uniform wind and wind with a lineal variation as function of time, of similar mean force. If we have, for

instance, a mean wind of $v = 10$ m/s at our disposal, varying from 3 to 17 m/s in a straight line - a case which does not occur in nature, - but is chosen for the sake of simplification - a velocity of $v' = 11.6$ m/s must be inserted in the calculation of power.

As $v' > v$, it may be supposed that gusts of wind might be desirable for a windmill. This is not the case, however, as the windmill could not follow such abrupt variations and would therefore be working at a loss under such conditions.

7. A windmill may be regulated for CONSTANT power L_0 in a variable wind. The question then arises as to what thrust the windmill encounters in a variable wind. If the coefficients β and ϵ designate the ratios of v^2 and α to v_0^2 and α_0 , the following relations are obtained:

$$\begin{aligned} L_0 &= \frac{1}{4} \rho v_0^3 F (1 + \alpha_0) (1 - \alpha_0^2) = L \\ &= \frac{1}{4} \rho \beta^{3/2} v_0^3 F (1 + \epsilon \alpha_0) (1 - \epsilon^2 \alpha_0^2) \end{aligned}$$

$$S_0 = \frac{1}{2} \rho v_0^2 F (1 - \alpha_0^2)$$

$$S = \frac{1}{2} \rho \beta v_0^2 F (1 - \epsilon^2 \alpha_0^2)$$

From this we obtain:

$$\frac{S}{S_0} = \left[\frac{(1 + \alpha_0) (1 - \epsilon \alpha_0)}{(1 - \alpha_0) (1 + \epsilon \alpha_0)} \right]^{1/3} = K$$

$$\beta = \left[\frac{(1 + \alpha_0) (1 - \alpha_0^3)}{(1 + \epsilon \alpha_0) (1 - \epsilon^3 \alpha_0^3)} \right]^{2/3}$$

and on eliminating ϵ :

$$\beta = \left[\frac{1 + \alpha_0 + \frac{\kappa^3}{2\kappa} (1 - \alpha_0)}{2\kappa} \right]^2$$

For an initial value of $\alpha_0 = 1/3$, the sweep of the values of ϵ and of the corresponding values of α are expressed in terms of β in Fig. 8. We know that the values of κ , and also the thrust exerted on the windmill, decrease when β increases; that is, with increased dynamic pressure. When the power is supplied with values of $\epsilon < 1$, the values of κ and the thrust increase when β increases; when $\epsilon > 1$, the values of κ and the thrust decrease when β increases.

The maximum value $\kappa = 1.26$ is attained for $\epsilon = 0$ when $\beta = 1.120$. The maximum power being utilized when $\alpha = 1/3$, - uniform power being assumed - $\beta < 1$ would not be possible. For values of $1 < \beta < 1.120$, it is calculated that two positions are possible, with varying thrust. As the upper position is situated on the curve branch resulting in an impossible stationary windmill, in which the velocity penetrating the windmill surface, $w = \frac{1}{2} v (1 + \alpha)$ diminishes, it causes a reduced number of revolutions and lower rotary moment of the wheel, with a determined coefficient of advance λ ; it may therefore be concluded that this curve branch does not actually come into question, and that with varying dynamic pressure the lower and declining curve branch

of κ must always be considered. This result is important for the "unloaded" windmill; it shows that increased thrust on the supports of a windmill cannot be disregarded, but that they are solely dependent on the maximum power utilized.

8. A work lately published in a well-known paper* should be compared with the ideal windmill theory here developed.

Professor BAUDISCH (Vienna) compares his theory with that of the water turbine and supposes the windmill to be working at a height of water due to the pressure and specific weight of the air. He makes a general difference between high, low, and normal pressure windmills. According to his theory, the air may undergo relative acceleration in the vanes, or it may remain at uniform velocity or be relatively retarded; finally, the cells may be influenced from without, in the axial direction or from within. BAUDISCH thereby obtains 27 different methods of possible construction. Thirteen of these only have, however, proved to be capable of construction, while one alone - a high or normal pressure turbine, with relative acceleration of the air in the wheel vanes and with interior admission, - to which he gives the preference. He further investigates the airstream conditions in front of, inside and behind the windmill and comes to the conclusion that the airflow behind

* "Theory of the Windmill," by Prof. Dr. Hans BAUDISCH, "Zeitschrift für das gesamte Turbinenwesen," 1917, pp.153 and 169. "Article on the Calculation of the Windmill," by Prof. Dr. BAUDISCH, in the Zeitschrift für das gesamte Turbinenwesen," 1920, pp. 125 and 136.

the windmill is almost exactly similar in height to that in front of it. The front part, as given by him, corresponds approximately to our representation of the airflow of the windmill (see Fig. 1). The drawing of the rear part is, however, inaccurate on the whole, so that reference should be made to Fig. 1 in place of it. He erroneously takes BENDEMANN'S view of the stationary propeller in his calculation of ratio of the diameters, which is in the present case a ratio of surfaces and should not be thus taken by BAUDISCH, the windmill being "in motion". BAUDISCH includes the viscosity of the air in his calculation, holding it responsible for the screen effect in front of the air wheel. In Karl SCHMID'S lecture* on the stationary propeller, the value of his factor of viscosity is inserted as for which we have not been able to find an adequate basis in this work.

We cannot adopt the views of BAUDISCH. The results of his calculations cannot be brought into agreement with the results of the general theory described above, and must therefore be rejected.

* "Zeitschrift für Flugtechnik und Motorluftschiffahrt," 1915.

VON W. HOFF - THEORY OF THE IDEAL AIRCRAFT

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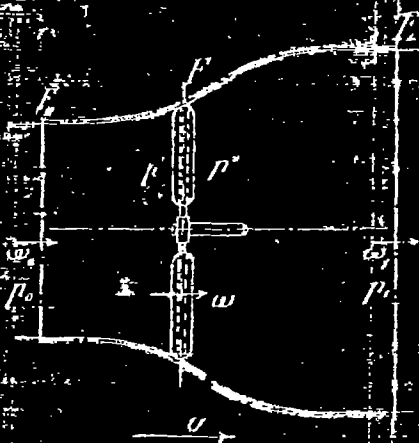


FIG. 1

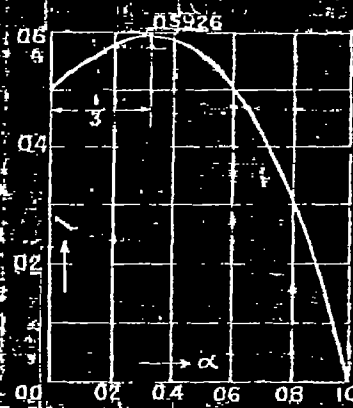


FIG. 2

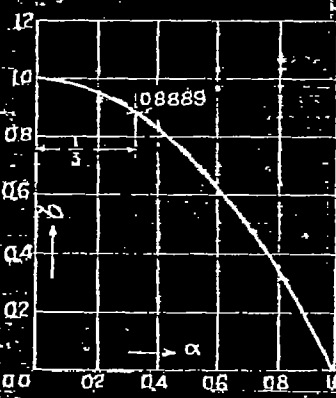


FIG. 3

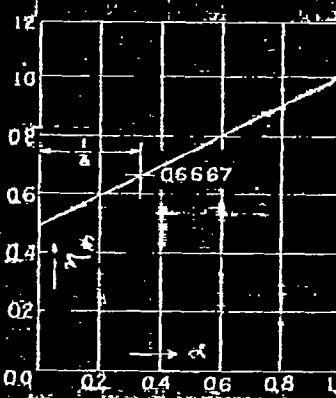


FIG. 4

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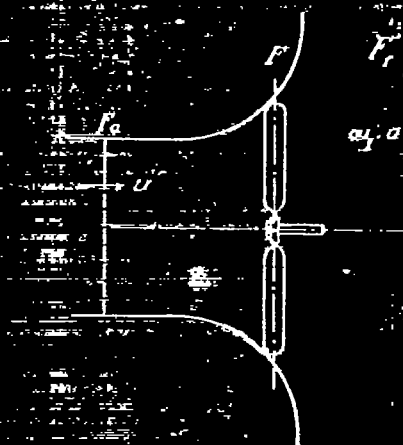


FIG. 5

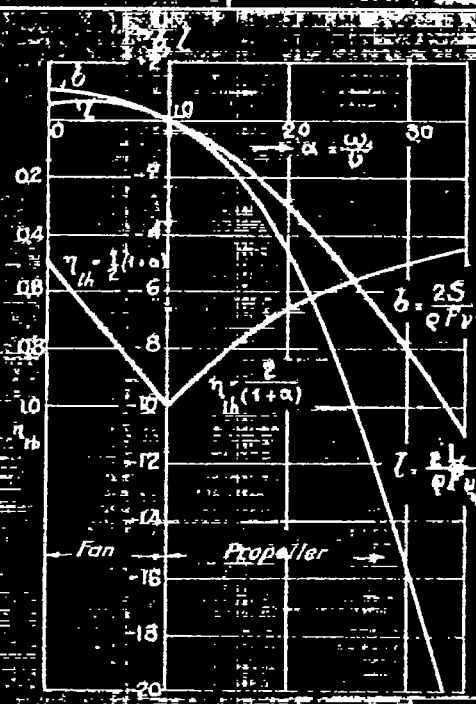


FIG. 1

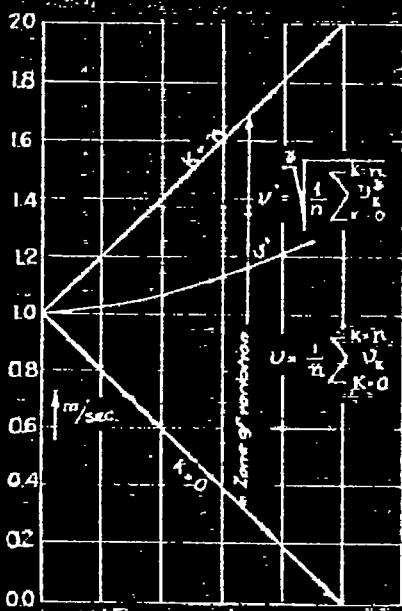


FIG. 7

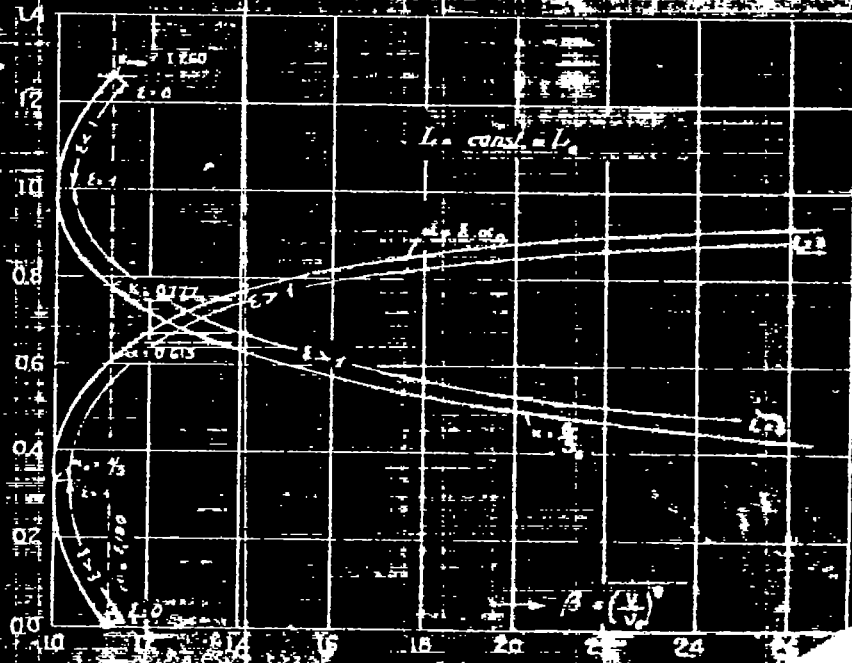


FIG. 2