Rocket Power Anomaly

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The Anomaly:

Imagine a rocket that accelerates from standstill while burning fuel at a constant rate. Intuitively we would think that the power applied to the rocket is constant. However this is not the case.

Let us say that the rocket engine produces a force F, and as a result the velocity of the rocket, v = dx/dt, increases with time. The definition of work done is force moved through a distance. Therefore the work done by the rocket motor at each instant t is given by Fdx. The power is therefore (Fdx)/dt. Since F is constant, the power delivered to the rocket is Fv. In other words, the power delivered to the rocket increases in proportion to its speed. How can this be if the rate of fuel burn is constant?

Analysis

Rocket acceleration

The rocket accelerates by pushing a mass of gas, known as the ejecta, out of the back. Let us define some quantities:
Initial mass of rocket = M
Instantaneous mass of rocket = m
Mass of ejecta during time t = dm
Velocity of ejecta with respect to the rocket = u
Rate of mass ejection = q = dm/dt

The force on the rocket at instant t is found by calculating the change of momentum of the ejecta. At the start of time interval dt, the ejecta has velocity v. At the end it has velocity (v-u)
The rate of change of momentum of the ejecta gives **F** = {v – (v-u)}dm/dt **= uq.**

The acceleration of the rocket can be calculated from this using the formula f = ma[i]. For a constant burn rate, the instantaneous mass of the rocket is **m** = M – t(dm/dt) **= M-qt.**

Rocket acceleration = uq/(M – tq)

Rocket Velocity

The speed of the rocket can be found by integrating this expression:

V = uq/(M-qt)dt

$V = -u.\ln(M-qt) + S$
Using the initial condition that when $t = 0$, $V = 0$, we find the integration constant $S = u*\ln(M)$
$V = u*\ln(M) – u*\ln(M-qt)$

Precise expression: $V = u*\ln(M/(M-qt))$______________________(1)

In the simplified case where the mass of the rocket is taken to be constant,

Rocket acceleration = uq/M

$V = uq/Mdt$

$V= uqt/M + S$

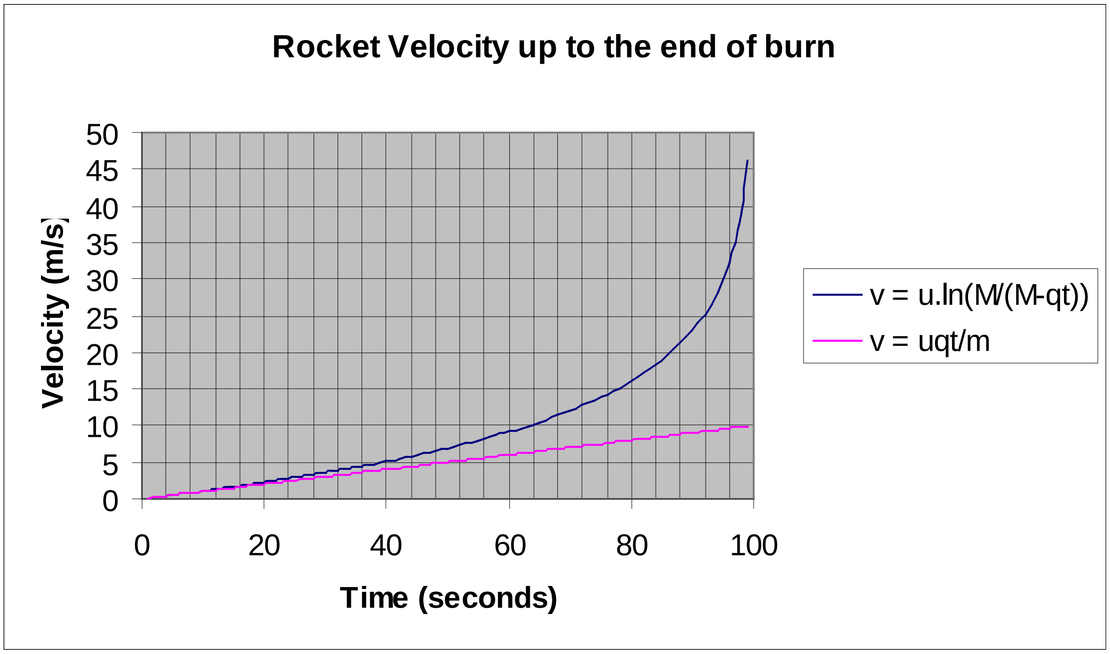
Using the initial condition that $v = 0$ when $t = 0$ we see that in this case $S = 0$.

Approximate expression: $V = uqt/M$________________________(2)

At first glance it might seem that (1) and (2) are entirely different equations. However they plot very similar curves when $qt << M$ (i.e. when the mass loss of the rocket is negligible)

The two curves are plotted below for the sample values
$M = 100$ kg
$q = 1$kg/s
$u = 10$ m/s

At that rate of burn, it should be obvious that the rocket will have run out of reaction mass after less than 100 seconds. The graph is only plotted for the first 99 seconds



So it can be seen that the two equations, although very different in appearance, give similar results at the start of the burn.

In the simplified equation the rocket speed never exceeds the relative speed of the ejecta, 10m/s. In the accurate equation the rocket speed exceeds the speed of the ejecta after 64 seconds. The speed then starts to increase rapidly as the rocket mass drops until, in the limit, we would have zero mass travelling at infinite velocity.

Rocket Power

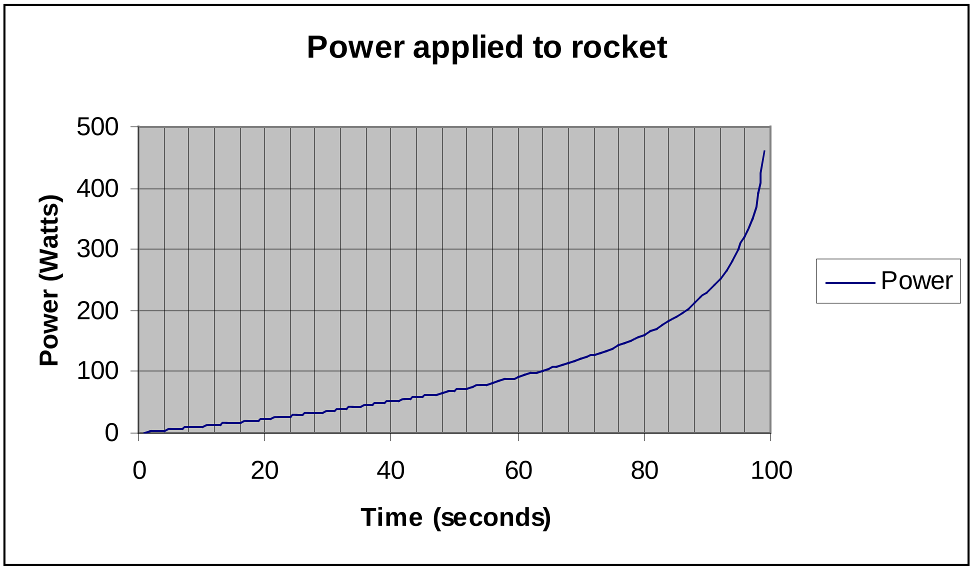
The power applied to the rocket by the rocket motor is given by force * velocity.

Power P uq * v

Since v = u*ln(M/(M-qt),

P = u^2q*ln(M/M-qt))

The power is plotted below:



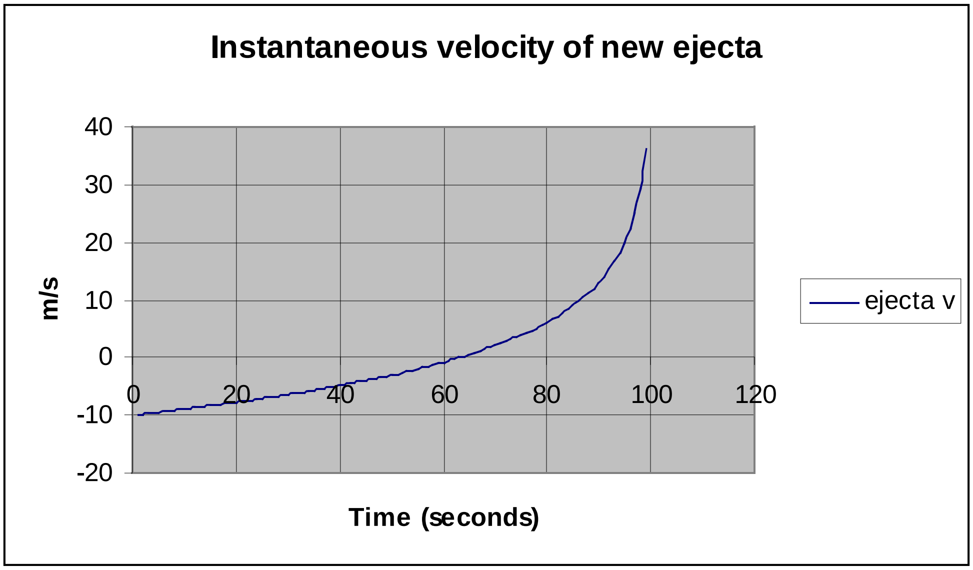
This is simply the graph of velocity scaled by the constant force (10 newtons in this example)

This is where the anomaly comes in. We know that if the rocket motor was held stationary, no power would be applied to the rocket. However the ejecta would be gaining kinetic energy at a constant rate. Every second, a mass of 1kg would be accelerated up to 10m/s giving it a kinetic energy of $1/2mv^2$ or 50 joules. We should therefore expect the power of the rocket to be a constant 50 watts. Why should it produce a different value of power if the rocket is allowed to move. In particular, how can it produce more power than 50 watts?

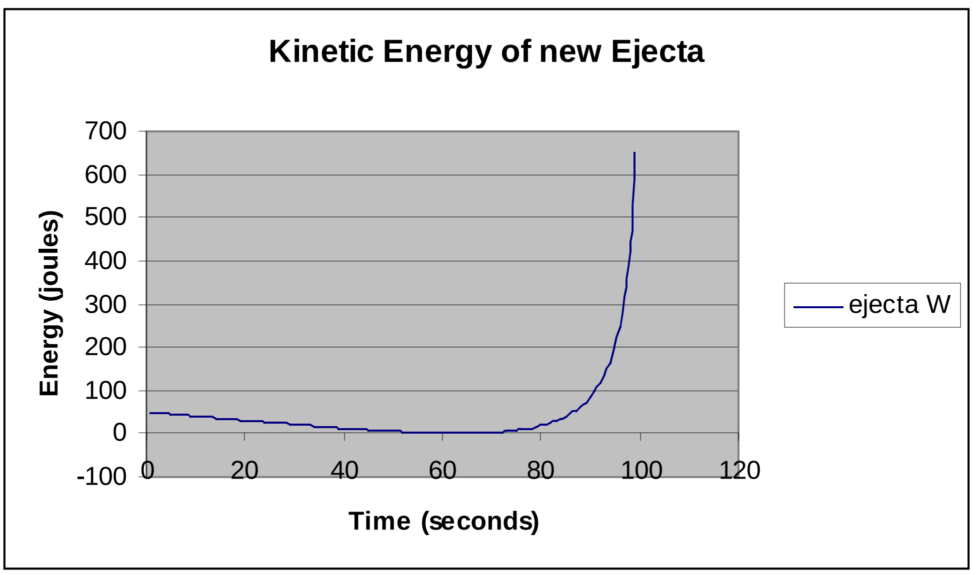
Energy transferred to the exhaust gases

Obviously some of the power of the rocket motor is being applied to the ejecta at the start of burn. We need to consider what is happening to the ejecta throughout the burn.

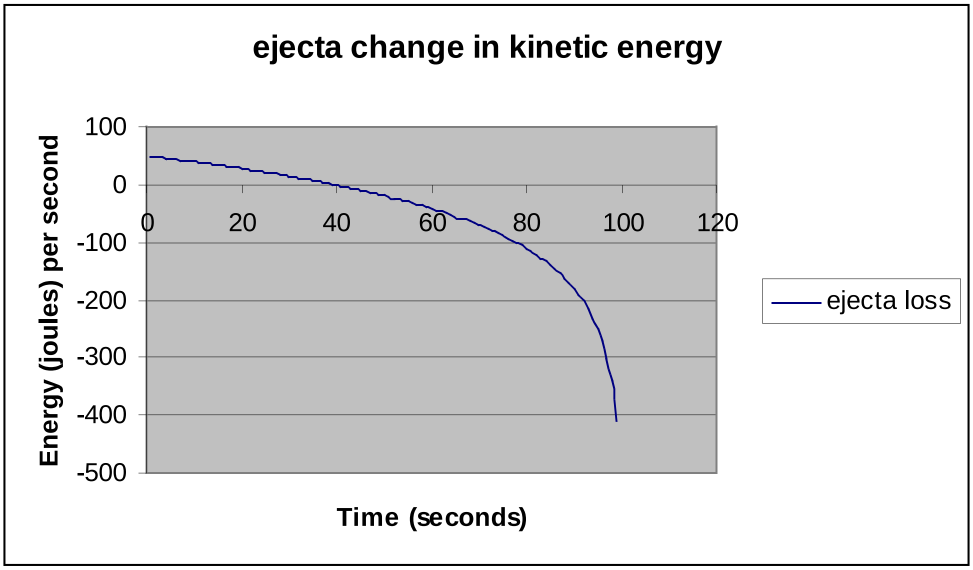
At each instant, a mass of 1kg will be ejected with a velocity equal to the rocket velocity – u. We can plot the instantaneous velocity of the ejecta and it will look like the instantaneous velocity of the rocket with a negative offset.



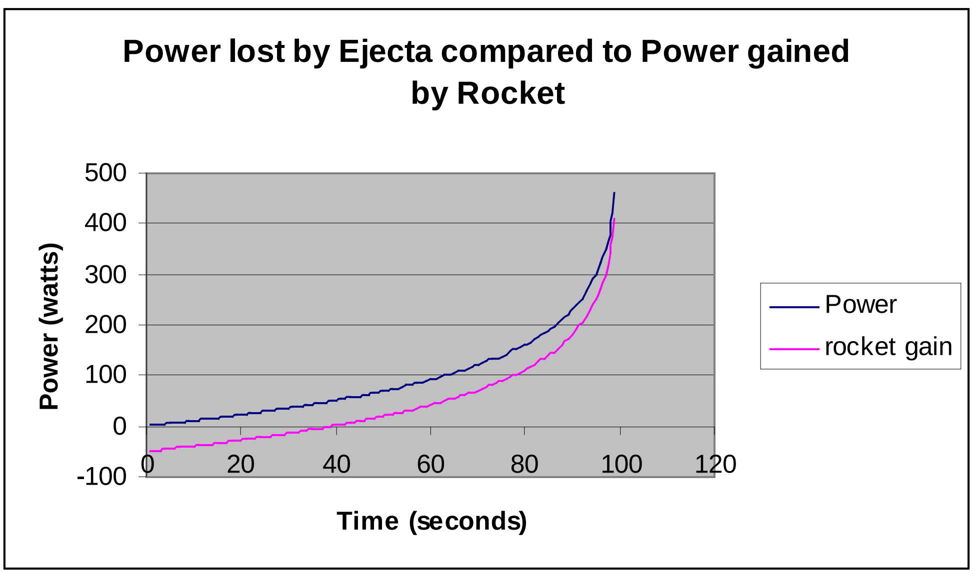
We can plot the kinetic energy of this new ejecta:



Finally we can plot the change in kinetic energy of the new ejecta compared to when it was sitting in the rocket. Prior to being blasted out of the rocket energy, it had a kinetic energy of $1/2v^2$ (The mass of the ejecta is 1kg a second so no term for mass is required.). After it has been ejected, it has a kinetic energy of $1/2(v-u)^2$ The graph below shows the change in kinetic energy of the ejecta as it is ejected for each second of the rocket's flight.



Notice that the change in kinetic energy has the units of power. If we take the power lost by the ejecta (the inverse of the graph above) and add to it the constant 50 watt power of the rocket motor, we should get the power supplied to the rocket. The graph below shows the power lost by the ejecta plotted on the same axes as the power gained by the rocket.



The difference between the two is the constant 50 watts provided by the rocket motor.

The Answer to the Conundrum

The solution to the anomaly can be put this way.

- The rocket motor produces a constant power.

- The power applied to the rocket increases as its velocity increases.
- The power applied to the rocket is the sum of the power from the rocket motor and the transfer of kinetic energy from the ejecta.

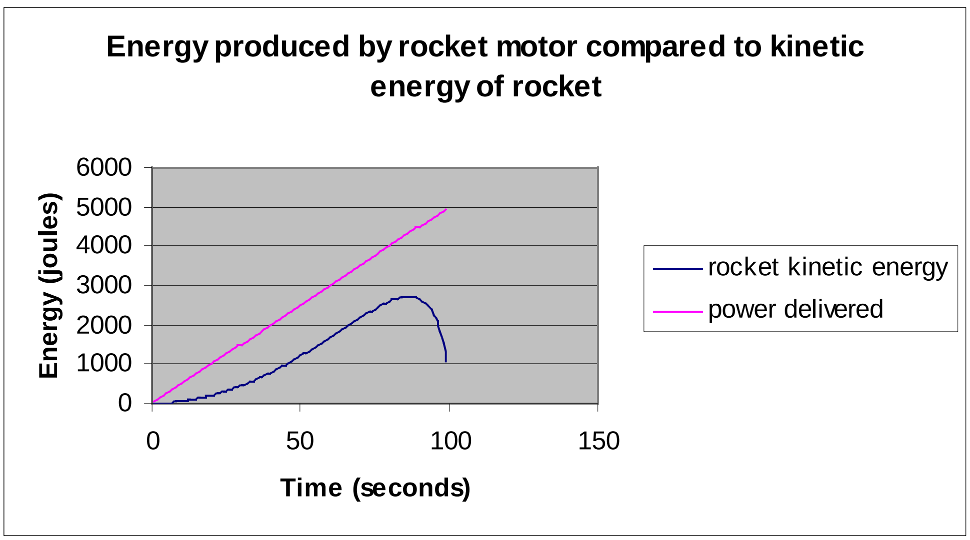
Loose End

Since the power applied to the rocket increases at a high rate, it is instructive to compare the total kinetic energy of the rocket to the total power supplied by the rocket.

The total power supplied by the rocket motor is the integral of the power and equals $\frac{1}{2}(dm/dt)u^2t$. In our example this is 50t.

The kinetic energy of the rocket is given by $1/2mv^2$
$= \frac{1}{2}(M-qt)[u*\ln(M/(M-qt))]^2$

This increases as the rocket gains speed but eventually decreases as the mass of the rocket starts to reduce.



At no time does the kinetic energy of the rocket exceed the energy delivered by the rocket motor. The energy produced by the rocket motor that does not find its way into the rocket is transferred to the ejecta.

[i] Note that it is sometimes maintained by SR experts that the only correct definition of force is rate of change of momentum because this is the only correct way to calculate the acceleration of a body that has non-constant mass. It is true that use of f = ma in SR is incorrect if f and m are defined as three vectors. However in the case of the rocket, where the mass of the rocket is decreasing, it is perfectly valid to use f = ma. The reason is that in the case of SR, the apparent change in mass is a result of the acceleration whereas in the case of the rocket the change of mass is the cause of the acceleration. The lost mass, the mass of the ejecta, is removed from the rocket before calculating the acceleration.