

# Introduction to High Energy Physics

Physics 85200

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## Conventions

- The Lorentz metric is  $g_{\mu\nu} = \text{diag}(+ - - -)$ .
- The totally antisymmetric tensor  $\epsilon_{\mu\nu\sigma\tau}$  satisfies  $\epsilon_{0123} = +1$ .
- We use a chiral basis for the Dirac matrices

$$\gamma^0 = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix} \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$$

$$\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} -\mathbf{1} & 0 \\ 0 & \mathbf{1} \end{pmatrix}$$

where the Pauli matrices are

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

- The quantum of electric charge is  $e = \sqrt{4\pi\alpha} > 0$ . I'll write the charge of the electron as  $eQ$  with  $Q = -1$ .
- Compared to Peskin & Schroeder we've flipped the signs of the gauge couplings ( $e \rightarrow -e$ ,  $g \rightarrow -g$ ) in all vertices and covariant derivatives. So for example in QED the covariant derivative is  $\mathcal{D}_\mu = \partial_\mu + ieQA_\mu$  and the electron – photon vertex is  $-ieQ\gamma^\mu$ . (This is a matter of convention because only  $e^2$  is observable. Our convention agrees with Quigg and is standard in non-relativistic quantum mechanics.)



## Useful formulas

Propagators:	$\frac{i}{p^2 - m^2}$	scalar
	$\frac{i(\not{p} + m)}{p^2 - m^2}$	spin-1/2
	$\frac{-ig_{\mu\nu}}{k^2}$	massless vector
	$\frac{-i(g_{\mu\nu} - k_\mu k_\nu / m^2)}{k^2 - m^2}$	massive vector

Vertex factors:	$\phi^4$ theory	appendix A
	spinor and scalar QED	appendix A
	QCD	chapter 10
	standard model	appendix E

Spin sums:	$\left. \begin{aligned} \sum_{\lambda} u(p, \lambda) \bar{u}(p, \lambda) &= \not{p} + m \\ \sum_{\lambda} v(p, \lambda) \bar{v}(p, \lambda) &= \not{p} - m \end{aligned} \right\}$	spin-1/2
	$\sum_i \epsilon_\mu^i \epsilon_\nu^{i*} = -g_{\mu\nu}$	massless vector (QED only)
	$\sum_i \epsilon_\mu^i \epsilon_\nu^{i*} = -g_{\mu\nu} + \frac{k_\mu k_\nu}{m^2}$	massive vector

In QCD in general one should only sum over physical gluon polarizations: see p. 113.

Trace formulas:

$\text{Tr} (\text{odd } \# \gamma\text{'s}) = 0$	$\text{Tr} ((\text{odd } \# \gamma\text{'s}) \gamma^5) = 0$
$\text{Tr} (\mathbf{1}) = 4$	$\text{Tr} (\gamma^5) = 0$
$\text{Tr} (\gamma^\mu \gamma^\nu) = 4g^{\mu\nu}$	$\text{Tr} (\gamma^\mu \gamma^\nu \gamma^5) = 0$
$\text{Tr} (\gamma^\alpha \gamma^\beta \gamma^\gamma \gamma^\delta) = 4(g^{\alpha\beta} g^{\gamma\delta} - g^{\alpha\gamma} g^{\beta\delta} + g^{\alpha\delta} g^{\beta\gamma})$	$\text{Tr} (\gamma^\alpha \gamma^\beta \gamma^\gamma \gamma^\delta \gamma^5) = 4i\epsilon^{\alpha\beta\gamma\delta}$

Decay rate  $1 \rightarrow 2 + 3$ :

In the center of mass frame

$$\Gamma = \frac{|\mathbf{p}|}{8\pi m^2} \langle |\mathcal{M}|^2 \rangle$$

Here  $\mathbf{p}$  is the spatial momentum of either outgoing particle and  $m$  is the mass of the decaying particle. If the final state has identical particles, divide the result by 2.

Cross section  $1 + 2 \rightarrow 3 + 4$ :

The center of mass differential cross section is

$$\left( \frac{d\sigma}{d\Omega} \right)_{\text{c.m.}} = \frac{1}{64\pi^2 s} \frac{|\mathbf{p}_3|}{|\mathbf{p}_1|} \langle |\mathcal{M}|^2 \rangle$$

where  $s = (p_1 + p_2)^2$  and  $|\mathbf{p}_1|$ ,  $|\mathbf{p}_3|$  are the magnitudes of the *spatial* 3-momenta. This expression is valid whether or not there are identical particles in the final state. However in computing a total cross section one should only integrate over inequivalent final configurations.

## Particle properties

Leptons, quarks and gauge bosons:

particle		charge	mass	lifetime / width	principal decays
Leptons	$\nu_e, \nu_\mu, \nu_\tau$	0	0	stable	—
	$e^-$	-1	0.511 MeV	stable	—
	$\mu^-$	-1	106 MeV	$2.2 \times 10^{-6}$ sec	$e^- \bar{\nu}_e \nu_\mu$
	$\tau^-$	-1	1780 MeV	$2.9 \times 10^{-13}$ sec	$\pi^- \pi^0 \nu_\tau, \mu^- \bar{\nu}_\mu \nu_\tau, e^- \bar{\nu}_e \nu_\tau$
Quarks	$u$	2/3	3 MeV	—	—
	$c$	2/3	1.3 GeV	—	—
	$t$	2/3	172 GeV	—	—
	$d$	-1/3	5 MeV	—	—
	$s$	-1/3	100 MeV	—	—
	$b$	-1/3	4.2 GeV	—	—
Gauge bosons	photon	0	0	stable	—
	$W^\pm$	$\pm 1$	80.4 GeV	2.1 GeV	$W^+ \rightarrow \ell^+ \nu_\ell, u \bar{d}, c \bar{s}$
	$Z$	0	91.2 GeV	2.5 GeV	$\ell^+ \ell^-, \nu \bar{\nu}, q \bar{q}$
	gluon	0	0	—	—

Pseudoscalar mesons (spin-0, odd parity):

meson	quark content	charge	mass	lifetime	principal decays
$\pi^\pm$	$u\bar{d}, d\bar{u}$	$\pm 1$	140 MeV	$2.6 \times 10^{-8}$ sec	$\pi^+ \rightarrow \mu^+ \nu_\mu$
$\pi^0$	$(u\bar{u} - d\bar{d})/\sqrt{2}$	0	135 MeV	$8.4 \times 10^{-17}$ sec	$\gamma\gamma$
$K^\pm$	$u\bar{s}, s\bar{u}$	$\pm 1$	494 MeV	$1.2 \times 10^{-8}$ sec	$K^+ \rightarrow \mu^+ \nu_\mu, \pi^+ \pi^0$
$K^0, \bar{K}^0$	$d\bar{s}, s\bar{d}$	0	498 MeV	—	—
$K_S^0$	$K^0, \bar{K}^0$ mix to	"	"	$9.0 \times 10^{-11}$ sec	$\pi^+ \pi^-, \pi^0 \pi^0$
$K_L^0$	form $K_S^0, K_L^0$	"	"	$5.1 \times 10^{-8}$ sec	$\pi^\pm e^\mp \nu_e, \pi^\pm \mu^\mp \nu_\mu, \pi\pi\pi$
$\eta$	$(u\bar{u} + d\bar{d} - 2s\bar{s})/\sqrt{6}$	0	548 MeV	$5.1 \times 10^{-19}$ sec	$\gamma\gamma, \pi^0 \pi^0 \pi^0, \pi^+ \pi^- \pi^0$
$\eta'$	$(u\bar{u} + d\bar{d} + s\bar{s})/\sqrt{3}$	0	958 MeV	$3.4 \times 10^{-21}$ sec	$\pi^+ \pi^- \eta, \pi^0 \pi^0 \eta, \rho^0 \gamma$

$$\begin{array}{l} \text{isospin multiplets:} \quad \begin{pmatrix} \pi^+ \\ \pi^0 \\ \pi^- \end{pmatrix} \quad \begin{pmatrix} K^+ \\ K^0 \end{pmatrix} \quad \begin{pmatrix} \bar{K}^0 \\ K^- \end{pmatrix} \quad \eta \quad \eta' \\ \text{strangeness:} \quad \quad \quad 0 \quad \quad 1 \quad \quad -1 \quad \quad 0 \quad 0 \end{array}$$

Vector mesons (spin-1):

meson	quark content	charge	mass	width	principal decays
$\rho$	$u\bar{d}, (u\bar{u} - d\bar{d})/\sqrt{2}, d\bar{u}$	+1, 0, -1	775 MeV	150 MeV	$\pi\pi$
$K^*$	$u\bar{s}, d\bar{s}, s\bar{d}, s\bar{u}$	+1, 0, 0, -1	892 MeV	51 MeV	$K\pi$
$\omega$	$(u\bar{u} + d\bar{d})/\sqrt{2}$	0	783 MeV	8.5 MeV	$\pi^+ \pi^- \pi^0$
$\phi$	$s\bar{s}$	0	1019 MeV	4.3 MeV	$K^+ K^-, K_L^0 K_S^0$

$$\begin{array}{l} \text{isospin multiplets:} \quad \begin{pmatrix} \rho^+ \\ \rho^0 \\ \rho^- \end{pmatrix} \quad \begin{pmatrix} K^{*+} \\ K^{*0} \end{pmatrix} \quad \begin{pmatrix} \bar{K}^{*0} \\ K^{*-} \end{pmatrix} \quad \omega \quad \phi \\ \text{strangeness:} \quad \quad \quad 0 \quad \quad 1 \quad \quad -1 \quad \quad 0 \quad 0 \end{array}$$

Spin-1/2 baryons:

baryon	quark content	charge	mass	lifetime	principal decays
$p$	$uud$	+1	938.3 MeV	stable	–
$n$	$udd$	0	939.6 MeV	886 sec	$pe^- \bar{\nu}_e$
$\Lambda$	$uds$	0	1116 MeV	$2.6 \times 10^{-10}$ sec	$p\pi^-, n\pi^0$
$\Sigma^+$	$uus$	+1	1189 MeV	$8.0 \times 10^{-11}$ sec	$p\pi^0, n\pi^+$
$\Sigma^0$	$uds$	0	1193 MeV	$7.4 \times 10^{-20}$ sec	$\Lambda\gamma$
$\Sigma^-$	$dds$	-1	1197 MeV	$1.5 \times 10^{-10}$ sec	$n\pi^-$
$\Xi^0$	$uss$	0	1315 MeV	$2.9 \times 10^{-10}$ sec	$\Lambda\pi^0$
$\Xi^-$	$dss$	-1	1322 MeV	$1.6 \times 10^{-10}$ sec	$\Lambda\pi^-$

$$\begin{array}{l} \text{isospin multiplets:} \quad \left( \begin{array}{c} p \\ n \end{array} \right) \quad \Lambda \quad \left( \begin{array}{c} \Sigma^+ \\ \Sigma^0 \\ \Sigma^- \end{array} \right) \quad \left( \begin{array}{c} \Xi^0 \\ \Xi^- \end{array} \right) \\ \text{strangeness:} \quad \quad \quad 0 \quad -1 \quad -1 \quad -2 \end{array}$$

Spin-3/2 baryons:

baryon	quark content	charge	mass	width / lifetime	principal decays
$\Delta$	$uuu, uud, udd, ddd$	+2, +1, 0, -1	1232 MeV	118 MeV	$p\pi, n\pi$
$\Sigma^*$	$uus, uds, dds$	+1, 0, -1	1387 MeV	39 MeV	$\Lambda\pi, \Sigma\pi$
$\Xi^*$	$uss, dss$	0, -1	1535 MeV	10 MeV	$\Xi\pi$
$\Omega^-$	$sss$	-1	1672 MeV	$8.2 \times 10^{-11}$ sec	$\Lambda K^-, \Xi^0\pi^-$

$$\begin{array}{l} \text{isospin multiplets:} \quad \left( \begin{array}{c} \Delta^{++} \\ \Delta^+ \\ \Delta^0 \\ \Delta^- \end{array} \right) \quad \left( \begin{array}{c} \Sigma^{*+} \\ \Sigma^{*0} \\ \Sigma^{*-} \end{array} \right) \quad \left( \begin{array}{c} \Xi^{*0} \\ \Xi^{*-} \end{array} \right) \quad \Omega^- \\ \text{strangeness:} \quad \quad \quad 0 \quad -1 \quad -2 \quad -3 \end{array}$$

The particle data book denotes strongly-decaying particles by giving their approximate mass in parenthesis, *e.g.* the  $\Sigma^*$  baryon is known as the  $\Sigma(1385)$ . The values listed for  $K^*$ ,  $\Sigma^*$ ,  $\Xi^*$  are for the state with charge  $-1$ .





# 1

## The particle zoo

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The observed interactions can be classified as strong, electromagnetic, weak and gravitational. Here are some typical decay processes:

strong:	$\Delta^0 \rightarrow p\pi^-$	lifetime $6 \times 10^{-24}$ sec
	$\rho^0 \rightarrow \pi^+\pi^-$	$4 \times 10^{-24}$ sec
electromag:	$\Sigma^0 \rightarrow \Lambda\gamma$	$7 \times 10^{-20}$ sec
	$\pi^0 \rightarrow \gamma\gamma$	$8 \times 10^{-17}$ sec
weak:	$\pi^- \rightarrow \mu^- \bar{\nu}_\mu$	$2.6 \times 10^{-8}$ sec
	$n \rightarrow pe^- \bar{\nu}_e$	15 minutes

The extremely short lifetime of the  $\Delta^0$  indicates that the decay is due to the strong force. Electromagnetic decays are generally slower, and weak decays are slower still. Gravity is so weak that it has no influence on observed particle physics (and will hardly be mentioned for the rest of this course).

The observed particles can be classified into

- hadrons: particles that interact strongly (as well as via the electromagnetic and weak forces). Hadrons can either carry integer spin ('mesons') or half-integer spin ('baryons'). Literally hundreds of hadrons have been detected: the mesons include  $\pi$ ,  $K$ ,  $\eta$ ,  $\rho$ , ... and the baryons include  $p$ ,  $n$ ,  $\Delta$ ,  $\Sigma$ ,  $\Lambda$ , ...
- charged leptons: these are spin-1/2 particles that interact via the electromagnetic and weak forces. Only three are known:  $e$ ,  $\mu$ ,  $\tau$ .
- neutral leptons (also known as neutrinos): spin-1/2 particles that only feel the weak force. Again only three are known:  $\nu_e$ ,  $\nu_\mu$ ,  $\nu_\tau$ .
- gauge bosons: spin-1 particles that carry the various forces (gluons for the

strong force, the photon for electromagnetism,  $W^\pm$  and  $Z$  for the weak force).

All interactions have to respect some familiar conservation laws, such as conservation of charge, energy, momentum and angular momentum. In addition there are some conservation laws that aren't so familiar. For example, consider the process

$$p \rightarrow e^+ \pi^0.$$

This process respects conservation of charge and angular momentum, and there is plenty of energy available for the decay, but it has never been observed. In fact as far as anyone knows the proton is stable (the lower bound on the proton lifetime is  $10^{31}$  years). How to understand this? Introduce a conserved additive quantum number, the 'baryon number'  $B$ , with  $B = +1$  for baryons,  $B = -1$  for antibaryons, and  $B = 0$  for everyone else. Then the proton (as the lightest baryon) is guaranteed to be absolutely stable.

There's a similar law of conservation of lepton number  $L$ . In fact, in the lepton sector, one can make a stronger statement. The muon is observed to decay weakly, via

$$\mu^- \rightarrow e^- \bar{\nu}_e \nu_\mu.$$

However the seemingly allowed decay

$$\mu^- \rightarrow e^- \gamma$$

has never been observed, even though it respects all the conservation laws we've talked about so far. To rationalize this we introduce separate conservation laws for electron number, muon number and tau number  $L_e$ ,  $L_\mu$ ,  $L_\tau$ . These are defined in the obvious way, for instance

$$\begin{aligned} L_e &= +1 && \text{for } e^- \text{ and } \nu_e \\ L_e &= -1 && \text{for } e^+ \text{ and } \bar{\nu}_e \\ L_e &= 0 && \text{for everyone else} \end{aligned}$$

Note that the observed decay  $\mu^- \rightarrow e^- \bar{\nu}_e \nu_\mu$  indeed respects all these conservation laws.

So far all the conservation laws we've introduced are exact (at least, no violation has ever been observed). But now for a puzzle. Consider the decay

$$K^+ \rightarrow \pi^+ \pi^0 \quad \text{observed with } \approx 20\% \text{ branching ratio}$$

The initial and final states are all strongly-interacting (hadronic), so you

might expect that this is a strong decay. However the lifetime of the  $K^+$  is  $10^{-8}$  sec, characteristic of a weak decay. To understand this Gell-Mann and Nishijima proposed to introduce another additive conserved quantum number, called  $S$  for ‘strangeness.’ One assigns some rather peculiar values, for example  $S = 0$  for  $p$  and  $\pi$ ,  $S = 1$  for  $K^+$  and  $K^0$ ,  $S = -1$  for  $\Lambda$  and  $\Sigma$ ,  $S = -2$  for  $\Xi$ . Strangeness is conserved by the strong force and by electromagnetism, but can be violated by weak interactions. The decay  $K^+ \rightarrow \pi^+\pi^0$  violates strangeness by one unit, so it must be a weak decay. If this seems too cheap I should mention that strangeness explains more than just kaon decays. For example it also explains why

$$\Lambda \rightarrow p\pi^-$$

is a weak process (lifetime  $2.6 \times 10^{-10}$  sec).

Now for another puzzle: there are some surprising degeneracies in the hadron spectrum. For example the proton and neutron are almost degenerate,  $m_p = 938.3$  MeV while  $m_n = 939.6$  MeV. Similarly  $m_{\Sigma^+} = 1189$  MeV while  $m_{\Sigma^0} = 1193$  MeV and  $m_{\Sigma^-} = 1197$  MeV. Another example is  $m_{\pi^\pm} = 140$  MeV and  $m_{\pi^0} = 135$  MeV. ( $\pi^+$  and  $\pi^-$  have exactly the same mass since they’re a particle / antiparticle pair.)

Back in 1932 Heisenberg proposed that we should regard the proton and neutron as two different states of a single particle, the “nucleon.”

$$|p\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad |n\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

This is very similar to the way we represent a spin-up electron and spin-down electron as being two different states of a single particle. Pushing this analogy further, Heisenberg proposed that the strong interactions are invariant under “isospin rotations” – the analog of invariance under ordinary rotations for ordinary angular momentum. Putting this mathematically, we postulate some isospin generators  $I_i$  that obey the same algebra as angular momentum, and that commute with the strong Hamiltonian.

$$[I_i, I_j] = i\epsilon_{ijk}I_k \quad [I_i, H_{\text{strong}}] = 0 \quad i, j, k \in \{1, 2, 3\}$$

We can group particles into isospin multiplets, for example the nucleon doublet

$$\begin{pmatrix} p \\ n \end{pmatrix}$$

has total isospin  $I = 1/2$ , while the  $\Sigma$ ’s and  $\pi$ ’s are grouped into isotriplets

with  $I = 1$ :

$$\begin{pmatrix} \Sigma^+ \\ \Sigma^0 \\ \Sigma^- \end{pmatrix} \qquad \begin{pmatrix} \pi^+ \\ \pi^0 \\ \pi^- \end{pmatrix}$$

Note that isospin is definitely not a symmetry of electromagnetism, since we're grouping together particles with different charges. It's also not a symmetry of the weak interactions, since for example the weak decay of the pion  $\pi^- \rightarrow \mu^- \bar{\nu}_\mu$  violates isospin. Rather the claim is that if we could “turn off” the electromagnetic and weak interactions then isospin would be an exact symmetry and the proton and neutron would be indistinguishable.<sup>†</sup> (For ordinary angular momentum, this would be like having a Hamiltonian that can be separated into a dominant rotationally-invariant piece plus a small non-invariant perturbation. If you like, the weak and electromagnetic interactions pick out a preferred direction in isospin space.)

At this point isospin might just seem like a convenient book-keeping device for grouping particles with similar masses. But you can test isospin in a number of non-trivial ways. One of the classic examples is pion – proton scattering. At center of mass energies around 1200 MeV scattering is dominated by the formation of an intermediate  $\Delta$  resonance.

$$\begin{aligned} \pi^+ p &\rightarrow \Delta^{++} \rightarrow \text{anything} \\ \pi^0 p &\rightarrow \Delta^+ \rightarrow \text{anything} \\ \pi^- p &\rightarrow \Delta^0 \rightarrow \text{anything} \end{aligned}$$

The pion has  $I = 1$ , the proton has  $I = 1/2$ , and the  $\Delta$  has  $I = 3/2$ . Now recall the Clebsch-Gordon coefficients for adding angular momentum ( $J = 1$ )  $\otimes$  ( $J = 1/2$ ) to get ( $J = 3/2$ ).

$$\text{notation: } |J, M\rangle = \sum_{m_1, m_2} C_{m_1, m_2}^{J, M} |J_1, m_1\rangle |J_2, m_2\rangle$$

$$|3/2, 3/2\rangle = |1, 1\rangle |1/2, 1/2\rangle$$

$$|3/2, 1/2\rangle = \frac{1}{\sqrt{3}} |1, 1\rangle |1/2, -1/2\rangle + \sqrt{\frac{2}{3}} |1, 0\rangle |1/2, 1/2\rangle$$

$$|3/2, -1/2\rangle = \sqrt{\frac{2}{3}} |1, 0\rangle |1/2, -1/2\rangle + \sqrt{\frac{1}{3}} |1, -1\rangle |1/2, 1/2\rangle$$

$$|3/2, -3/2\rangle = |1, -1\rangle |1/2, -1/2\rangle$$

<sup>†</sup> As we'll see isospin is also violated by quark masses. To the extent that one regards quark masses as a part of the strong interactions, one should say that even  $H_{\text{strong}}$  has a small isospin-violating component.

From this we can conclude that the amplitudes stand in the ratio

$$\langle \pi^+ p | H_{\text{strong}} | \Delta^{++} \rangle : \langle \pi^0 p | H_{\text{strong}} | \Delta^+ \rangle : \langle \pi^- p | H_{\text{strong}} | \Delta^0 \rangle = 1 : \sqrt{\frac{2}{3}} : \sqrt{\frac{1}{3}}$$

This is either obvious (if you don't think about it too much), or a special case of the Wigner-Eckart theorem.<sup>†</sup> Anyhow you're supposed to prove it on the homework.

Since we don't care what the  $\Delta$  decays to, and since decay rates go like the  $|\cdot|^2$  of the matrix element (Fermi's golden rule), we conclude that near 1200 MeV the cross sections should satisfy

$$\sigma(\pi^+ p \rightarrow X) : \sigma(\pi^0 p \rightarrow X) : \sigma(\pi^- p \rightarrow X) = 1 : \frac{2}{3} : \frac{1}{3}$$

This fits the data quite well. See the plots on the next page.

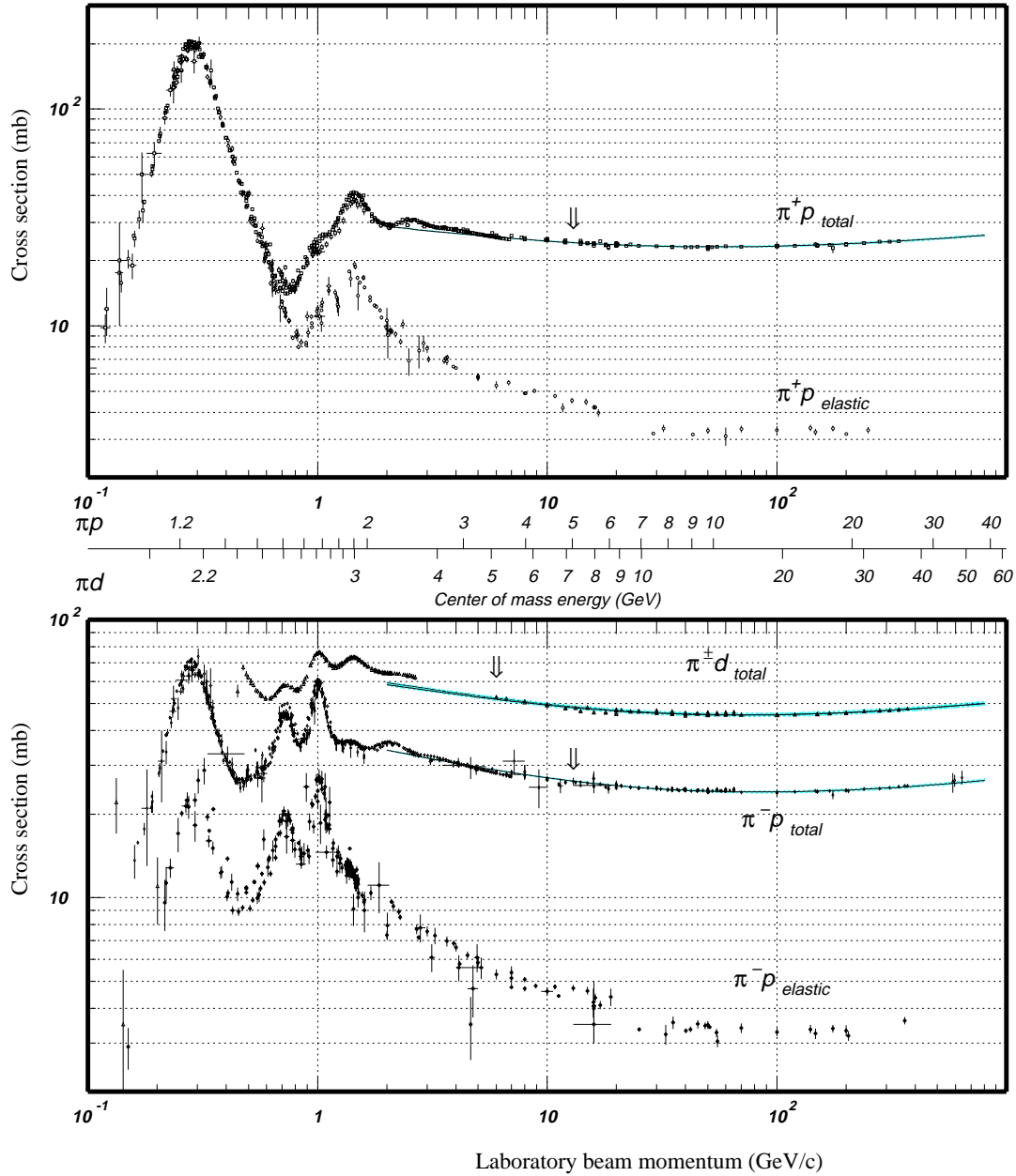
The conservation laws we've discussed in this chapter are summarized in the following table.

conservation law	strong	EM	weak
energy $E$	✓	✓	✓
charge $Q$	✓	✓	✓
baryon # $B$	✓	✓	✓
lepton #'s $L_e, L_\mu, L_\tau$	✓	✓	✓
strangeness $S$	✓	✓	×
isospin $I$	✓	×	×

## References

The basic forces, particles and conservation laws are discussed in the introductory chapters of Griffiths and Halzen & Martin. Isospin is discussed in section 4.5 of Griffiths.

<sup>†</sup> For the general formalism see Sakurai, *Modern Quantum Mechanics* p. 239.



**Figure 39.14:** Total and elastic cross sections for  $\pi^\pm p$  and  $\pi^\pm d$  (total only) collisions as a function of laboratory beam momentum and total center-of-mass energy. Corresponding computer-readable data files may be found at <http://pdg.lbl.gov/xsect/contents.html> (Courtesy of the COMPAS Group, IHEP, Protvino, Russia, August 2001.)

**Exercises****1.1 Decays of the spin-3/2 baryons**

The spin-3/2 baryons in the “baryon decuplet” ( $\Delta$ ,  $\Sigma^*$ ,  $\Xi^*$ ,  $\Omega$ ) are all unstable. The  $\Delta$ ,  $\Sigma^*$  and  $\Xi^*$  decay strongly, with a lifetime  $\sim 10^{-23}$  sec. The  $\Omega$ , however, decays weakly (lifetime  $\sim 10^{-10}$  sec). To see why this is, consider the following decays:

1.  $\Delta^+ \rightarrow p\pi^0$
2.  $\Sigma^{*-} \rightarrow \Lambda\pi^-$
3.  $\Sigma^{*-} \rightarrow \Sigma^-\pi^0$
4.  $\Xi^{*-} \rightarrow \Xi^-\pi^0$
5.  $\Omega^- \rightarrow \Xi^-\bar{K}^0$
6.  $\Omega^- \rightarrow \Lambda K^-$

- (i) For each of these decays, which (if any) of the conservation laws we discussed are violated? You should check  $E, Q, B, S, I$ .
- (ii) Based on this information, which (if any) interaction is responsible for these decays?

**1.2 Decays of the spin-1/2 baryons**

Most of the spin-1/2 baryons in the “baryon octet” (nucleon,  $\Lambda$ ,  $\Sigma$ ,  $\Xi$ ) decay weakly to another spin-1/2 baryon plus a pion. The two exceptions are the  $\Sigma^0$  (which decays electromagnetically) and the neutron (which decays weakly to  $pe^-\bar{\nu}_e$ ). To see why this is, consider the following decays:

1.  $\Xi^- \rightarrow \Lambda K^-$
2.  $\Xi^- \rightarrow \Lambda\pi^-$
3.  $\Sigma^- \rightarrow \Lambda\pi^-$
4.  $\Sigma^- \rightarrow n\pi^-$
5.  $\Sigma^0 \rightarrow \Lambda\gamma$
6.  $\Lambda \rightarrow n\pi^0$
7.  $n \rightarrow p\pi^-$
8.  $n \rightarrow pe^-\bar{\nu}_e$

- (i) For each of these decays, which (if any) of the conservation laws we discussed are violated? You should check  $E, Q, B, L, S, I$ .

- (ii) Based on this information, which (if any) interaction is responsible for these decays? You can assume a photon indicates an electromagnetic process, while a neutrino indicates a weak process.

### 1.3 Meson decays

Consider the following decays:

1.  $\pi^- \rightarrow e^- \bar{\nu}_e$
2.  $\pi^0 \rightarrow \gamma\gamma$
3.  $K^- \rightarrow \pi^- \pi^0$
4.  $K^- \rightarrow \mu^- \bar{\nu}_\mu$
5.  $\eta \rightarrow \gamma\gamma$
6.  $\eta \rightarrow \pi^0 \pi^0 \pi^0$
7.  $\rho^- \rightarrow \pi^- \pi^0$

- (i) For each of these decays, which (if any) of the conservation laws we discussed are violated? You should check  $E, Q, B, L, S, I$ .
- (ii) Based on this information, which interaction is responsible for these decays? You can assume a photon indicates an electromagnetic process, while a neutrino indicates a weak process.
- (iii) Look up the lifetimes of these particles. Do they fit with your expectations?

### 1.4 Isospin and the $\Delta$ resonance

Suppose the strong interaction Hamiltonian is invariant under an  $SU(2)$  isospin symmetry,  $[H_{\text{strong}}, \mathbf{I}] = 0$ . By inserting suitable isospin raising and lowering operators  $I_\pm = I_1 \pm iI_2$  show that (up to possible phases)

$$\frac{1}{\sqrt{3}} \langle \Delta^{++} | H_{\text{strong}} | \pi^+ p \rangle = \frac{1}{\sqrt{2}} \langle \Delta^+ | H_{\text{strong}} | \pi^0 p \rangle = \langle \Delta^0 | H_{\text{strong}} | \pi^- p \rangle.$$

### 1.5 Decay of the $\Xi^*$

The  $\Xi^*$  baryon decays primarily to  $\Xi + \pi$ . For a neutral  $\Xi^*$  there are two possible decays:

$$\Xi^{*0} \rightarrow \Xi^0 \pi^0$$

$$\Xi^{*0} \rightarrow \Xi^- \pi^+$$

Use isospin to predict the branching ratios.



1.6  $\Delta I = 1/2$  rule

The  $\Lambda$  baryon decays weakly to a nucleon plus a pion. The Hamiltonian responsible for the decay is

$$H = \frac{1}{\sqrt{2}} G_F \bar{u} \gamma^\mu (1 - \gamma^5) d \bar{s} \gamma_\mu (1 - \gamma^5) u + \text{c.c.}$$

This operator changes the strangeness by  $\pm 1$  and the  $z$  component of isospin by  $\mp 1/2$ . It can be decomposed  $H = H_{3/2} + H_{1/2}$  into pieces which carry total isospin  $3/2$  and  $1/2$ , since  $\bar{u} \gamma^\mu (1 - \gamma^5) d$  transforms as  $|1, -1\rangle$  and  $\bar{s} \gamma_\mu (1 - \gamma^5) u$  transforms as  $|1/2, 1/2\rangle$ . The (theoretically somewhat mysterious) “ $\Delta I = 1/2$  rule” states that the  $I = 1/2$  part of the Hamiltonian dominates.

- (i) Use the  $\Delta I = 1/2$  rule to relate the matrix elements  $\langle p \pi^- | H | \Lambda \rangle$  and  $\langle n \pi^0 | H | \Lambda \rangle$ .
- (ii) Predict the corresponding branching ratios for  $\Lambda \rightarrow p \pi^-$  and  $\Lambda \rightarrow n \pi^0$ .

The PDG gives the branching ratios  $\Lambda \rightarrow p \pi^- = 63.9\%$  and  $\Lambda \rightarrow n \pi^0 = 35.8\%$ .

## 2

### Flavor $SU(3)$ and the eightfold way

Physics 85200

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Last time we encountered a zoo of conservation laws, some of them only approximate. In particular baryon number was exactly conserved, while strangeness was conserved by the strong and electromagnetic interactions, and isospin was only conserved by the strong force. Our goal for the next few weeks is to find some order in this madness. Since the strong interactions seem to be the most symmetric, we're going to concentrate on them. Ultimately we're going to combine  $B$ ,  $S$  and  $\mathbf{I}$  and understand them as arising from a symmetry of the strong interactions.

At this point, it's not clear how to get started. One idea, which several people explored, is to extend  $SU(2)$  isospin symmetry to a larger symmetry – that is, to group different isospin multiplets together. However if you list the mesons with odd parity, zero spin, and masses less than 1 GeV

$\pi^\pm, \pi^0$	135 to 140 MeV
$K^\pm, K^0, \bar{K}^0$	494 to 498 MeV
$\eta$	548 MeV
$\eta'$	958 MeV

it's not at all obvious how (or whether) these particles should be grouped together. Somehow this didn't stop Gell-Mann, who in 1961 proposed that  $SU(2)$  isospin symmetry should be extended to an  $SU(3)$  flavor symmetry.

#### 2.1 Tensor methods for $SU(N)$ representations

$SU(2)$  is familiar from angular momentum, but before we can go any further we need to know something about  $SU(3)$  and its representations. It turns out that we might as well do the general case of  $SU(N)$ .

First some definitions; if you need more of an introduction to group theory see section 4.1 of Cheng & Li.  $SU(N)$  is the group of  $N \times N$  unitary matrices with unit determinant,

$$UU^\dagger = \mathbf{1}, \quad \det U = 1.$$

We're interested in *representations* of  $SU(N)$ . This just means we want a vector space  $V$  and a rule that associates to every  $U \in SU(N)$  a linear operator  $\mathcal{D}(U)$  that acts on  $V$ . The key property that makes it a representation is that the multiplication rule is respected,

$$\mathcal{D}(U_1)\mathcal{D}(U_2) = \mathcal{D}(U_1U_2)$$

(on the left I'm multiplying the linear operators  $\mathcal{D}(U_1)$  and  $\mathcal{D}(U_2)$ , on the right I'm multiplying the two unitary matrices  $U_1$  and  $U_2$ ).

One representation of  $SU(N)$  is almost obvious from the definition: just set  $\mathcal{D}(U) = U$ . That is, let  $U$  itself act on an  $N$ -component vector  $\mathbf{z}$ .

$$\mathbf{z} \rightarrow U\mathbf{z}$$

This is known as the fundamental or  $N$ -dimensional representation of  $SU(N)$ .

Another representation is not quite so obvious: set  $\mathcal{D}(U) = U^*$ . That is, let the complex conjugate matrix  $U^*$  act on an  $N$ -component vector.

$$\mathbf{z} \rightarrow U^*\mathbf{z}$$

In this case we need to check that the multiplication law is respected; fortunately

$$\mathcal{D}(U_1)\mathcal{D}(U_2) = U_1^*U_2^* = (U_1U_2)^* = \mathcal{D}(U_1U_2).$$

This is known as the antifundamental or conjugate representation of  $SU(N)$ . It's often denoted  $\bar{\mathbf{N}}$ .

At this point it's convenient to introduce some index notation. We'll write the fundamental representation as acting on a vector with an upstairs index,

$$z^i \rightarrow U^i_j z^j$$

where  $U^i_j \equiv (ij \text{ element of } U)$ . We'll write the conjugate representation as acting on a vector with a downstairs index,

$$z_i \rightarrow U_i^j z_j$$

where  $U_i^j \equiv (ij \text{ element of } U^*)$ . We take complex conjugation to exchange upstairs and downstairs indices.

Given any number of fundamental and conjugate representations we can

multiply them together (take a tensor product, in mathematical language). For example something like  $x_i y^j z^k$  would transform under  $SU(N)$  according to

$$x_i y^j z^k \rightarrow U_i^l U^j_m U^k_n x_l y^m z^n.$$

Such a tensor product representation is in general *reducible*. This just means that the linear operators  $\mathcal{D}(U)$  can be simultaneously block-diagonalized, for all  $U \in SU(N)$ .

We're interested in breaking the tensor product up into its irreducible pieces. To accomplish this we can make use of the following  $SU(N)$ -invariant tensors:

$\delta^i_j$	Kronecker delta
$\epsilon_{i_1 \dots i_N}$	totally antisymmetric Levi-Civita
$\epsilon^{i_1 \dots i_N}$	another totally antisymmetric Levi-Civita

Index positions are very important here: for example  $\delta_{ij}$  with both indices downstairs is *not* an invariant tensor. It's straightforward to check that these tensors are invariant; it's mostly a matter of unraveling the notation. For example

$$\delta^i_j \rightarrow U^i_k U_j^l \delta^k_l = U^i_k U_j^k = U^i_k (U^*)^j_k = U^i_k (U^\dagger)^k_j = (U U^\dagger)^i_j = \delta^i_j$$

One can also check

$$\epsilon^{i_1 \dots i_N} \rightarrow U^{i_1}_{j_1} \dots U^{i_N}_{j_N} \epsilon^{j_1 \dots j_N} = \det U \epsilon^{i_1 \dots i_N} = \epsilon^{i_1 \dots i_N}$$

with a similar argument for  $\epsilon_{i_1 \dots i_N}$ .

Decomposing tensor products is useful in its own right, but it also provides a way to make irreducible representations of  $SU(N)$ . The procedure for making irreducible representations is

1. Start with some number of fundamental and antifundamental representations: say  $m$  fundamentals and  $n$  antifundamentals.
2. Take their tensor product.
3. Use the invariant tensors to break the tensor product up into its irreducible pieces.

The claim (which I won't try to prove) is that by repeating this procedure for all values of  $m$  and  $n$ , one obtains all of the irreducible representations of  $SU(N)$ .<sup>†</sup>

<sup>†</sup> Life isn't so simple for other groups.

## 2.2 $SU(2)$ representations

To get oriented let's see how this works for  $SU(2)$ . All the familiar results about angular momentum can be obtained using these tensor methods.

First of all, what are the irreducible representations of  $SU(2)$ ? Let's start with a general tensor  $T_{j_1 j_2 \dots j_n}^{i_1 i_2 \dots i_m}$ . Suppose we've already classified all representations with fewer than  $k = m + n$  indices; we want to identify the new irreducible representations that appear at rank  $k$ . First note that by contracting with  $\epsilon^{ij}$  we can move all indices upstairs; for example starting with an antifundamental  $z_i$  we can construct

$$\epsilon^{ij} z_j$$

which transforms as a fundamental. So we might as well just look at tensors with upstairs indices:  $T^{i_1 \dots i_k}$ . We can break  $T$  up into two pieces, which are either symmetric or antisymmetric under exchange of  $i_1$  with  $i_2$ :

$$T^{i_1 \dots i_k} = \frac{1}{2} (T^{i_1 i_2 \dots i_k} + T^{i_2 i_1 \dots i_k}) + \frac{1}{2} (T^{i_1 i_2 \dots i_k} - T^{i_2 i_1 \dots i_k}) .$$

The antisymmetric piece can be written as  $\epsilon^{i_1 i_2}$  times a tensor of lower rank (with  $k - 2$  indices). So let's ignore the antisymmetric piece, and just keep the piece which is symmetric on  $i_1 \leftrightarrow i_2$ . If you repeat this symmetrization / antisymmetrization process on all pairs of indices you'll end up with a tensor  $S^{i_1 \dots i_k}$  that is symmetric under exchange of any pair of indices. At this point the procedure stops: there's no way to further decompose  $S$  using the invariant tensors.

So we've learned that  $SU(2)$  representations are labeled by an integer  $k = 0, 1, 2, \dots$ ; in the  $k^{\text{th}}$  representation a totally symmetric tensor with  $k$  indices transforms according to

$$S^{i_1 \dots i_k} \rightarrow U^{i_1}_{j_1} \dots U^{i_k}_{j_k} S^{j_1 \dots j_k}$$

To figure out the dimension of the representation (meaning the dimension of the vector space) we need to count the number of independent components of such a tensor. This is easy, the independent components are

$$S^{1 \dots 1}, S^{1 \dots 12}, S^{1 \dots 122}, \dots, S^{2 \dots 2}$$

so the dimension of the representation is  $k + 1$ .

In fact we have just recovered all the usual representations of angular momentum. To make this more apparent we need to change terminology a bit: we define the spin by  $j \equiv k/2$ , and call  $S^{i_1 \dots i_{2j}}$  the spin- $j$  representation.

The dimension of the representation has the familiar form,  $\dim(j) = 2j + 1$ . Some examples:

representation	tensor	name	dimension	spin
$k = 0$		trivial	1	$j = 0$
$k = 1$	$z^i$	fundamental	2	$j = 1/2$
$k = 2$	$S^{ij}$	symmetric tensor	3	$j = 1$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

All the usual results about angular momentum can be reproduced in tensor language. For example, consider addition of angular momentum. With two spin-1/2 particles the total angular momentum is either zero or one. To see this in tensor language one just multiplies two fundamental representations and then decomposes into irreducible pieces:

$$z^i w^j = \frac{1}{2} (z^i w^j + z^j w^i) + \frac{1}{2} (z^i w^j - z^j w^i)$$

The first term is symmetric so it transforms in the spin one representation. The second term is antisymmetric so it's proportional to  $\epsilon^{ij}$  and hence has spin zero.

### 2.3 $SU(3)$ representations

Now let's see how things work for  $SU(3)$ . Following the same procedure, we start with an arbitrary tensor  $T_{j_1 j_2 \dots}^{i_1 i_2 \dots}$ . First let's work on the upstairs indices. Decompose

$$T_{j_1 j_2 \dots j_n}^{i_1 i_2 \dots i_m} = (\text{piece that's symmetric on } i_1 \leftrightarrow i_2) \\ + (\text{piece that's antisymmetric on } i_1 \leftrightarrow i_2).$$

The antisymmetric piece can be written as

$$\epsilon^{i_1 i_2 k} \tilde{T}_{k j_1 j_2 \dots j_n}^{i_3 i_4 \dots i_m}$$

in terms of a tensor  $\tilde{T}$  with lower rank (two fewer upstairs indices but one more downstairs index). So we can forget about the piece that's antisymmetric on  $i_1 \leftrightarrow i_2$ . Repeating this procedure for all upstairs index pairs, we end up with a tensor that's totally symmetric on the upstairs indices. Following a similar procedure with the help of  $\epsilon_{ijk}$ , we can further restrict attention to tensors that are symmetric under exchange of any two downstairs indices.

For  $SU(3)$  there's one further decomposition we can make. We can write

$$S_{j_1 j_2 \dots j_n}^{i_1 i_2 \dots i_m} = \frac{1}{3} \delta_{j_1}^{i_1} S_{k j_2 \dots j_n}^{k i_2 \dots i_m} + \tilde{S}_{j_1 j_2 \dots j_n}^{i_1 i_2 \dots i_m}$$

where  $\tilde{S}$  is traceless on it's first indices,  $\tilde{S}_{k j_2 \dots j_n}^{k i_2 \dots i_m} = 0$ . Throwing out the trace part, and repeating this procedure on all upstairs / downstairs index pairs, we see that  $SU(3)$  irreps act on tensors  $T_{j_1 j_2 \dots j_n}^{i_1 i_2 \dots i_m}$  that are

- symmetric under exchange of any two upstairs indices
- symmetric under exchange of any two downstairs indices
- traceless, meaning if you contract any upstairs index with any downstairs index you get zero

This is known as the  $\begin{pmatrix} m \\ n \end{pmatrix}$  representation of  $SU(3)$ . Some examples:

representation	tensor	name	dimension	notation
$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$		trivial	1	<b>1</b>
$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$	$z^i$	fundamental	3	<b>3</b>
$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$	$z_i$	conjugate	3	<b><math>\bar{3}</math></b>
$\begin{pmatrix} 2 \\ 0 \end{pmatrix}$	$S^{ij}$	symmetric tensor	6	<b>6</b>
$\begin{pmatrix} 1 \\ 1 \end{pmatrix}$	$T_j^i$	adjoint	8	<b>8</b>
$\begin{pmatrix} 0 \\ 2 \end{pmatrix}$	$S_{ij}$	symmetric tensor	6	<b><math>\bar{6}</math></b>
$\begin{pmatrix} 3 \\ 0 \end{pmatrix}$	$S^{ijk}$	symmetric tensor	10	<b>10</b>
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Using these methods we can reduce product representations (the  $SU(3)$  analog of adding angular momentum). For example, to reduce the product **3**  $\otimes$  **3** we can write

$$z^i w^j = \frac{1}{2} (z^i w^j + z^j w^i) + \frac{1}{2} \epsilon^{ijk} v_k$$

where  $v_k = \epsilon_{klm} z^l v^m$ . In terms of representations this means

$$\mathbf{3} \otimes \mathbf{3} = \mathbf{6} \oplus \bar{\mathbf{3}}.$$

As another example, consider  $\mathbf{3} \otimes \bar{\mathbf{3}}$ .

$$\begin{aligned} z^i w_j &= \left( z^i w_j - \frac{1}{3} \delta_j^i z^k w_k \right) + \frac{1}{3} \delta_j^i z^k w_k \\ \Rightarrow \quad \mathbf{3} \otimes \bar{\mathbf{3}} &= \mathbf{8} \oplus \mathbf{1} \end{aligned}$$

Finally, let's do  $\mathbf{6} \otimes \mathbf{3}$ .

$$\begin{aligned} S^{ij} z^k &= \frac{1}{3} \left( S^{ij} z^k + S^{jk} z^i + S^{ki} z^j \right) + \frac{2}{3} S^{ij} z^k - \frac{1}{3} S^{jk} z^i - \frac{1}{3} S^{ki} z^j \\ &= \frac{1}{3} S^{ijk} + \frac{1}{3} T_l^i \epsilon^{ljk} + \frac{1}{3} T_l^j \epsilon^{lik} \end{aligned}$$

where  $S^{ijk} = S^{ij} z^k + (\text{cyclic perms})$  is in the  $\begin{pmatrix} 3 \\ 0 \end{pmatrix}$ , and  $T_l^i = \epsilon_{lmn} S^{im} z^n$

is in the  $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ . That is, we've found that

$$\mathbf{6} \otimes \mathbf{3} = \mathbf{10} \oplus \mathbf{8}.$$

If you want to keep going, it makes sense to develop some machinery to automate these calculations – but fortunately, this is all we'll need.

## 2.4 The eightfold way

Finally, some physics. Gell-Mann and Ne'eman proposed that the strong interactions have an  $SU(3)$  symmetry, and that all light hadrons should be grouped into  $SU(3)$  multiplets. As we've seen, all  $SU(3)$  multiplets can be built up starting from the  $\mathbf{3}$  and  $\bar{\mathbf{3}}$ . So at least as a mnemonic it's convenient to think in terms of elementary quarks and antiquarks

$$\begin{aligned} q &= \begin{pmatrix} u \\ d \\ s \end{pmatrix} && \text{in } \mathbf{3} \\ \bar{q} &= \begin{pmatrix} \bar{u} \\ \bar{d} \\ \bar{s} \end{pmatrix} && \text{in } \bar{\mathbf{3}} \end{aligned}$$

Here I'm embedding the  $SU(2)$  isospin symmetry inside  $SU(3)$  via

$$\begin{pmatrix} U & 0 \\ 0 & 1 \end{pmatrix} \in SU(3).$$



I'm also going to be associating one unit of strangeness with  $\bar{s}$ .<sup>†</sup> That is, in terms of isospin / strangeness the  $\mathbf{3}$  of  $SU(3)$  decomposes as

$$\mathbf{3} = \mathbf{2}_0 \oplus \mathbf{1}_{-1}.$$

(On the left hand side we have an  $SU(3)$  representation, on the right hand side I'm labeling  $SU(2)$  representations by their dimension and putting strangeness in the subscript.) The idea here is that (although they're both exact symmetries of the strong force) isospin is a better approximate symmetry than  $SU(3)_{\text{flavor}}$ , so isospin multiplets will be more nearly degenerate in mass than  $SU(3)$  multiplets.

All mesons are supposed to be quark – antiquark states. In terms of  $SU(3)$  representations we have  $\mathbf{3} \otimes \bar{\mathbf{3}} = \mathbf{8} \oplus \mathbf{1}$ , so mesons should be grouped into octets (hence the name “eightfold way”) and singlets. Further decomposing in terms of isospin and strangeness

$$(\mathbf{2}_0 \oplus \mathbf{1}_{-1}) \otimes (\mathbf{2}_0 \oplus \mathbf{1}_{+1}) = (\mathbf{2} \otimes \mathbf{2})_0 \oplus \mathbf{2}_1 \oplus \mathbf{2}_{-1} \oplus \mathbf{1}_0 = \mathbf{3}_0 \oplus \mathbf{1}_0 \oplus \mathbf{2}_1 \oplus \mathbf{2}_{-1} \oplus \mathbf{1}_0$$

That is, we should get

an isospin triplet with strangeness = 0	$\pi^+, \pi^0, \pi^-$
an isospin doublet with strangeness = +1	$K^+, K^0$
an isospin doublet with strangeness = -1	$\bar{K}^0, K^-$
two singlets with strangeness = 0	$\eta, \eta'$

Not bad!

The baryons are supposed to be 3-quark states. In terms of  $SU(3)$  representations we have  $\mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3} = (\mathbf{6} \oplus \bar{\mathbf{3}}) \otimes \mathbf{3} = \mathbf{10} \oplus \mathbf{8} \oplus \mathbf{8} \oplus \mathbf{1}$  so we get decuplets, octets and singlets. As an example, let's decompose the decuplet in terms of isospin and strangeness. Recall that the  $\mathbf{10}$  is a symmetric 3-index tensor so

$$\begin{aligned} & [(\mathbf{2}_0 \oplus \mathbf{1}_{-1}) \otimes (\mathbf{2}_0 \oplus \mathbf{1}_{-1}) \oplus (\mathbf{2}_0 \oplus \mathbf{1}_{-1})]_{\text{symmetrized}} \\ &= (\mathbf{2} \otimes \mathbf{2} \otimes \mathbf{2})_{\text{symmetrized}, 0} \oplus (\mathbf{2} \otimes \mathbf{2})_{\text{symmetrized}, -1} \oplus \mathbf{2}_{-2} \oplus \mathbf{1}_{-3} \\ &= \mathbf{4}_0 \oplus \mathbf{3}_{-1} \oplus \mathbf{2}_{-2} \oplus \mathbf{1}_{-3} \end{aligned}$$

(It's very convenient to think about the symmetrized  $SU(2)$  products in tensor language.) That is, we should get

<sup>†</sup> When strangeness was first introduced people didn't know about quarks. They gave the  $K^+$  strangeness +1, but it turns out the  $K^+$  contains an  $\bar{s}$  quark. Sorry about that.

isospin 3/2 with strangeness = 0	$\Delta^{++}, \Delta^+, \Delta^0, \Delta^-$
isospin 1 with strangeness = -1	$\Sigma^{*+}, \Sigma^{*0}, \Sigma^{*-}$
isospin 1/2 with strangeness = -2	$\Xi^{*0}, \Xi^{*-}$
isospin 0 with strangeness = -3	$\Omega^-$

One can't help but be impressed.

## 2.5 Symmetry breaking by quark masses

Having argued that hadrons should be grouped into  $SU(3)$  multiplets, we'd now like to understand the  $SU(3)$  breaking effects that give rise to the (rather large) mass splittings observed within each multiplet. It might seem hopeless to understand  $SU(3)$  breaking at this point, since we've argued that so many things (electromagnetism, weak interactions) violate  $SU(3)$ . But fortunately there are some  $SU(3)$  breaking effects – namely quark mass terms – which are easy to understand and are often the dominant source of  $SU(3)$  breaking.

The idea is to take quarks seriously as elementary particles, and to introduce a collection of Dirac spinor fields to describe them.

$$\psi = \begin{pmatrix} u \\ d \\ s \end{pmatrix}$$

Here  $\psi$  is a 3-component vector in flavor space; each entry in  $\psi$  is a 4-component Dirac spinor. Although we don't know the full Lagrangian for the strong interactions, we'd certainly expect it to include kinetic terms for the quarks.

$$\mathcal{L}_{\text{strong}} = \mathcal{L}_{\text{kinetic}} + \cdots \quad \mathcal{L}_{\text{kinetic}} = \bar{\psi} i \gamma^\mu \partial_\mu \psi$$

The quark kinetic terms are invariant under  $SU(3)$  transformations  $\psi \rightarrow U\psi$ . We're going to assume that all terms in  $\mathcal{L}_{\text{strong}}$  have this symmetry.

Now let's consider some possible  $SU(3)$  breaking terms. One fairly obvious possibility is to introduce mass terms for the quarks.<sup>†</sup>

$$\mathcal{L}_{SU(3)\text{-breaking}} = \mathcal{L}_{\text{mass}} + \cdots$$

<sup>†</sup> In the old days people took the strong interactions to be exactly  $SU(3)$  invariant, as we did above. They regarded mass terms as separate  $SU(3)$ -breaking terms in the Lagrangian. These days one tends to think of quark masses as part of the strong interactions, and regard  $\mathcal{L}_{\text{mass}}$  as an  $SU(3)$ -violating part of the strong interactions.

$$\mathcal{L}_{\text{mass}} = -\bar{\psi} M \psi \quad M = \begin{pmatrix} m_u & 0 & 0 \\ 0 & m_d & 0 \\ 0 & 0 & m_s \end{pmatrix}$$

These mass terms are, in general, not  $SU(3)$ -invariant. Rather the pattern of  $SU(3)$  breaking depends on the quark masses. The discussion is a bit simpler if we include the symmetry of multiplying  $\psi$  by an overall phase, that is, if we consider  $\psi \rightarrow U\psi$  with  $U \in U(3)$ .

- $m_u = m_d = m_s \quad \Rightarrow \quad U(3)$  is a valid symmetry
- $m_u = m_d \neq m_s \quad \Rightarrow \quad U(3)$  broken to  $U(2) \times U(1)$
- $m_u, m_d, m_s$  all distinct  $\Rightarrow \quad U(3)$  broken to  $U(1)^3$

In the first case we'd have a flavor  $SU(3)$  symmetry plus an additional  $U(1)$  corresponding to baryon number. In the second (most physical) case we'd have an isospin  $SU(2)$  symmetry acting on  $\begin{pmatrix} u \\ d \end{pmatrix}$  plus two additional  $U(1)$ 's which correspond to (linear combinations of) baryon number and strangeness. In the third case we'd have three  $U(1)$  symmetries corresponding to upness, downness and strangeness.

One can say this in a slightly fancier way: the  $SU(3)$  breaking pattern is determined by the eigenvalues of the quark mass matrix. To see this suppose we started with a general mass matrix  $M$  that isn't necessarily diagonal.  $M$  has to be Hermitian for the Lagrangian to be real, so we can write

$$M = U \begin{pmatrix} m_u & 0 & 0 \\ 0 & m_d & 0 \\ 0 & 0 & m_s \end{pmatrix} U^\dagger \quad m_u \leq m_d \leq m_s$$

for some  $U \in SU(3)$ . Then an  $SU(3)$  transformation of the quark fields  $\psi \rightarrow U\psi$  will leave  $\mathcal{L}_{\text{strong}}$  invariant and will bring the quark mass matrix to a diagonal form. But having chosen to diagonalize the mass matrix in this way, one is no longer free to make  $SU(3)$  transformations with off-diagonal entries unless some of the eigenvalues of  $M$  happen to coincide.

In the real world isospin  $SU(2)$  is a much better symmetry than flavor  $SU(3)$ . It's tempting to try to understand this as a consequence of having  $m_u \approx m_d \ll m_s$ . How well does this work? Let's look at the spin-3/2 baryon decuplet. Recall that this has the isospin / strangeness decomposition

$$\begin{array}{ll}
I = \frac{3}{2} & S = 0 & \left( \begin{array}{l} \Delta^{++} = uuu \\ \Delta^+ = uud \\ \Delta^0 = udd \\ \Delta^- = ddd \end{array} \right) \\
I = 1 & S = -1 & \left( \begin{array}{l} \Sigma^{*+} = uus \\ \Sigma^{*0} = uds \\ \Sigma^{*-} = dds \end{array} \right) \\
I = \frac{1}{2} & S = -2 & \left( \begin{array}{l} \Xi^{*0} = uss \\ \Xi^{*-} = dss \end{array} \right) \\
I = 0 & S = -3 & \left( \begin{array}{l} \Omega^- = sss \end{array} \right)
\end{array}$$

Denoting

$m_0$  = (common mass arising from strong interactions)

$$m_u \approx m_d \equiv m_{u,d}$$

we'd predict

$$m_\Delta = m_0 + 3m_{u,d}$$

$$m_{\Sigma^*} = m_0 + 2m_{u,d} + m_s$$

$$m_{\Xi^*} = m_0 + m_{u,d} + 2m_s$$

$$m_\Omega = m_0 + 3m_s$$

Although we can't calculate  $m_0$ , there is a prediction we can make: mass splittings between successive rows in the table should roughly equal, given by  $m_s - m_{u,d}$ . Indeed

$$m_{\Sigma^*} - m_\Delta = 155 \text{ MeV}$$

$$m_{\Xi^*} - m_{\Sigma^*} = 148 \text{ MeV}$$

$$m_\Omega - m_{\Xi^*} = 137 \text{ MeV}$$

(equal to within roughly  $\pm 5\%$ ). This suggests that most  $SU(3)$  breaking is indeed due to the strange quark mass.<sup>†</sup>

One comment: you might think you could incorporate the charm quark into this scheme by extending Gell-Mann's  $SU(3)$  to an  $SU(4)$  flavor symmetry. In principle this is possible, but in practice it's not useful: the charm

<sup>†</sup> Note that the mass splittings originate from the traceless part of the mass matrix, which transforms in the  $\mathbf{8}$  of  $SU(3)$ . To be fair, any term in the Hamiltonian that transforms like

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \in \mathbf{8}$$

will give rise to the observed pattern of mass splittings, so really what we've shown is that quark masses are a natural source for such a term.

quark mass is so large that it can't be treated as a small perturbation of the strong interactions.

## 2.6 Multiplet mixing

At this point you might think that  $SU(3)$  completely accounts for the spectrum of hadrons. To partially dispel this notion let's look at the light vector (spin-1) mesons, which come in an isotriplet  $(\rho^+, \rho^0, \rho^-)$ , two isodoublets  $(K^{*+}, K^{*0})$ ,  $(\bar{K}^{*0}, K^{*-})$  and two isosinglets  $\omega$ ,  $\phi$ .

At first sight everything is fine. We'd expect to find the  $SU(3)$  quantum numbers  $\mathbf{3} \otimes \mathbf{\bar{3}} = \mathbf{8} \oplus \mathbf{1}$ , or in terms of isospin and strangeness  $\mathbf{3}_0 \oplus \mathbf{2}_1 \oplus \mathbf{2}_{-1} \oplus \mathbf{1}_0 \oplus \mathbf{1}_0$ . It's tempting to assign the flavor wavefunctions

$$\begin{aligned}\rho^+, \rho^0, \rho^- &= u\bar{d}, \frac{1}{\sqrt{2}}(d\bar{d} - u\bar{u}), -d\bar{u} \\ K^{*+}, K^{*0} &= u\bar{s}, d\bar{s} \\ \bar{K}^{*0}, K^{*-} &= s\bar{d}, -s\bar{u} \\ \omega &= \frac{1}{\sqrt{6}}(u\bar{u} + d\bar{d} - 2s\bar{s}) \\ \phi &= \frac{1}{\sqrt{3}}(u\bar{u} + d\bar{d} + s\bar{s})\end{aligned}\tag{2.1}$$

Here we're identifying the  $\omega$  with the  $I = 0$  state in the octet and taking  $\phi$  to be an  $SU(3)$  singlet. Given our model for  $SU(3)$  breaking by quark masses we'd expect

$$\begin{aligned}m_\rho &\approx m_8 + 2m_{u,d} \\ m_{K^*} &\approx m_8 + m_{u,d} + m_s \\ m_\omega &\approx m_8 + \frac{1}{3} \cdot 2m_{u,d} + \frac{2}{3} \cdot 2m_s \\ m_\phi &\approx m_1 + \frac{2}{3} \cdot 2m_{u,d} + \frac{1}{3} \cdot 2m_s\end{aligned}$$

Here  $m_8$  ( $m_1$ ) is the contribution to the octet (singlet) mass arising from strong interactions. We've used the fact that according to (2.1) the  $\omega$ , for example, spends 1/3 of its time as a  $u\bar{u}$  or  $d\bar{d}$  pair and the other 2/3 as an  $s\bar{s}$  pair. It follows from these equations that  $m_\omega = \frac{4}{3}m_{K^*} - \frac{1}{3}m_\rho$ , but this prediction doesn't fit the data:  $m_\omega = 783 \text{ MeV}$  while  $\frac{4}{3}m_{K^*} - \frac{1}{3}m_\rho = 931 \text{ MeV}$ .

Rather than give up on  $SU(3)$ , Sakurai pointed out that – due to  $SU(3)$  breaking – states in the octet and singlet can mix. In particular we should

allow for mixing between the two isosinglet states (isospin is a good enough symmetry that multiplets with different isospins don't seem to mix):

$$\begin{pmatrix} |\omega\rangle \\ |\phi\rangle \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} |8\rangle \\ |1\rangle \end{pmatrix}$$

Here  $\theta$  is a mixing angle which relates the mass eigenstates  $|\omega\rangle, |\phi\rangle$  to the states with definite  $SU(3)$  quantum numbers introduced above:

$$\begin{aligned} |8\rangle &\equiv \frac{1}{\sqrt{6}}(|u\bar{u}\rangle + |d\bar{d}\rangle - 2|s\bar{s}\rangle) \\ |1\rangle &\equiv \frac{1}{\sqrt{3}}(|u\bar{u}\rangle + |d\bar{d}\rangle + |s\bar{s}\rangle) \end{aligned}$$

The mass we calculated above can be identified with the expectation value of the Hamiltonian in the octet state,  $\langle 8|H|8\rangle = 931 \text{ MeV}$ . On the other hand  $\langle 8|H|8\rangle = (\cos\theta\langle\omega| - \sin\theta\langle\phi|)H(\cos\theta|\omega\rangle - \sin\theta|\phi\rangle) = m_\omega \cos^2\theta + m_\phi \sin^2\theta$ . This allows us to calculate the mixing angle

$$\sin\theta = \sqrt{\frac{\langle 8|H|8\rangle - m_\omega}{m_\phi - m_\omega}} = \sqrt{\frac{931 \text{ MeV} - 783 \text{ MeV}}{1019 \text{ MeV} - 783 \text{ MeV}}} = 0.79$$

which fixes the flavor wavefunctions

$$\begin{aligned} |\omega\rangle &= 0.999\frac{1}{\sqrt{2}}(|u\bar{u}\rangle + |d\bar{d}\rangle) - 0.04|s\bar{s}\rangle \\ |\phi\rangle &= 0.999|s\bar{s}\rangle + 0.04\frac{1}{\sqrt{2}}(|u\bar{u}\rangle + |d\bar{d}\rangle). \end{aligned}$$

The  $\omega$  has very little strange quark content, while  $\phi$  is almost pure  $s\bar{s}$ . When combined with the OZI rule<sup>†</sup> this explains why the  $\phi$  decays predominantly to strange particles, unlike the  $\omega$  which decays primarily to pions:

$$\begin{array}{ll} \phi \rightarrow K^+ K^-, K^0 \bar{K}^0 & 83\% \text{ branching ratio} \\ \omega \rightarrow \pi^+ \pi^- \pi^0 & 89\% \text{ branching ratio} \end{array}$$

It also explains why the  $\phi$  lives longer than the  $\omega$ , even though there's more phase space available for its decay:

$$\begin{array}{ll} \phi \text{ lifetime} & 1.5 \times 10^{-22} \text{ sec} \\ \omega \text{ lifetime} & 0.8 \times 10^{-22} \text{ sec} \end{array}$$

I hope this illustrates some of the limitations of flavor  $SU(3)$ . Along these lines it's worth mentioning that the spectrum of light scalar (as opposed to

<sup>†</sup> see Cheng & Li p. 121

pseudoscalar) mesons is quite poorly understood, both theoretically and experimentally. One recent attempt at clarification is hep-ph/0204205.

## References

Cheng & Li is pretty good. For an introduction to group theory see section 4.1. Tensor methods are developed in section 4.3 and applied to the hadron spectrum in section 4.4. For a more elementary discussion see sections 5.8 and 5.9 of Griffiths. Symmetry breaking by quark masses is discussed by Cheng & Li on p. 119;  $\omega/\phi$  mixing is on p. 120. For a classic treatment of the whole subject see Sidney Coleman, *Aspects of symmetry*, chapter 1.

## Exercises

### 2.1 Casimir operator for $SU(2)$

A symmetric tensor with  $n$  indices provides a representation of  $SU(2)$  with spin  $s = n/2$ . In this representation the  $SU(2)$  generators can be taken to be

$$J_i = \frac{1}{2} \sigma_i \otimes \mathbf{1} \otimes \cdots \otimes \mathbf{1} + \cdots + \mathbf{1} \otimes \cdots \otimes \mathbf{1} \otimes \frac{1}{2} \sigma_i$$

where  $\sigma_i$  are the Pauli matrices. (There are  $n$  terms in this expression; in the  $k^{th}$  term the Pauli matrices act on the  $k^{th}$  index of the tensor.) The  $SU(2)$  Casimir operator is  $J^2 = \sum_i J_i J_i$ . Show that  $J^2$  has the expected eigenvalue in this representation.

### 2.2 Flavor wavefunctions for the baryon octet

The baryon octet can be represented as a 3-index tensor  $B^{ijk} = T^i_l \epsilon^{ljk}$  where  $T^i_l$  is traceless. For example, in a basis  $u = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ ,

$$d = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad s = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \text{the matrix } T = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ gives the}$$

flavor wavefunction of a proton  $u(ud - du)$ . Work out the flavor wavefunctions of the remaining members of the baryon octet. The hard part is getting the  $\Sigma^0$  and  $\Lambda$  right; you'll need to take linear combinations which have the right isospin.

### 2.3 Combining flavor + spin wavefunctions for the baryon octet

You might object to the octet wavefunctions worked out in problem 2.2 on the grounds that they don't respect Fermi statistics. For spin-1/2 baryons we can represent the spin of the baryon using a vector  $v^a$   $a = 1, 2$  which transforms in the **2** of the  $SU(2)$  angular momentum group.

- (i) Write down a 3-index tensor that gives the spin wavefunction for the (spin-1/2) quarks that make up the baryon. ( $v^a$  is the analog of  $T^i_l$  in problem 2.2. I'm asking you to find the analog of  $B^{ijk}$ .)
- (ii) Show how to combine your flavor and spin wavefunctions to make a state that is totally *symmetric* under exchange of any two quarks. It has to be totally symmetric so that, when combined with a totally antisymmetric color wavefunction, we get something that respects Fermi statistics.
- (iii) Suppose the quarks have no orbital angular momentum (as is usually the case in the ground state). Can you make an octet of baryons with spin 3/2?

### 2.4 Mass splittings in the baryon octet

In class we discussed a model for  $SU(3)$  breaking based on non-degenerate quark masses. Use this model to predict

$$m_\Lambda \approx m_\Sigma \approx \frac{m_\Xi + m_N}{2}$$

where  $m_N$  is the nucleon mass. To what accuracy are these relations actually satisfied?

### 2.5 Electromagnetic decays of the $\Sigma^*$

The up and down quarks have different electric charges, so electromagnetic interactions violate the isospin  $SU(2)$  subgroup of  $SU(3)$ . However the down and strange quarks have identical electric charges. This means that electromagnetism respects a *different*  $SU(2)$  subgroup of  $SU(3)$ , sometimes called  $U$ -spin, that acts on the quarks as

$$\begin{pmatrix} u \\ d \\ s \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 \\ 0 & U \end{pmatrix} \begin{pmatrix} u \\ d \\ s \end{pmatrix}.$$

Use this to show that the electromagnetic decay  $\Sigma^{*-} \rightarrow \Sigma^- \gamma$  is forbidden but that  $\Sigma^{*+} \rightarrow \Sigma^+ \gamma$  is allowed.



## 3

### Quark properties

Physics 85200

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#### 3.1 Quark properties

Quarks must have some unusual properties, if you take them seriously as elementary particles.

First, isolated quarks have never been observed. To patch this up we'll simply postulate 'quark confinement': the idea that quarks are always permanently bound inside mesons or baryons.

Second, quarks must have unusual (fractional!) electric charges.

$$\begin{aligned}\Delta^{++} \sim uuu &\Rightarrow Q_u = 2/3 \\ \Delta^- \sim ddd &\Rightarrow Q_d = -1/3 \\ \Omega^- \sim sss &\Rightarrow Q_s = -1/3\end{aligned}$$

There's nothing wrong with fractional charges, of course – it's just that they're a little unexpected.

Third, quarks are presumably spin-1/2 Dirac fermions. To see this note that baryons have half-integer spins and are supposed to be  $qqq$  bound states. The simplest possibility is to imagine that the quarks themselves carry spin 1/2. Then by adding the spin angular momenta of the quarks we can make

$$\begin{aligned}\text{mesons with spins} & \quad \frac{1}{2} \otimes \frac{1}{2} = 1 \oplus 0 \\ \text{baryons with spins} & \quad \frac{1}{2} \otimes \frac{1}{2} \otimes \frac{1}{2} = \frac{3}{2} \oplus \frac{1}{2} \oplus \frac{1}{2}\end{aligned}$$

You can make hadrons with even larger spins if you give the quarks some orbital angular momentum.

At this point there's a puzzle with Fermi statistics. Consider the combined

flavor and spin wavefunction for a  $\Delta^{++}$  baryon with spin  $s_z = 3/2$ .

$$|\Delta^{++} \text{ with } s_z = 3/2\rangle = |u \uparrow, u \uparrow, u \uparrow\rangle$$

The state is symmetric under exchange of any two quarks, in violation of Fermi statistics.

To rescue the quark model Nambu proposed that quarks carry an additional ‘color’ quantum number, associated with a new  $SU(3)$  symmetry group denoted  $SU(3)_{\text{color}}$ . This is in addition to the flavor and spin labels we’ve already talked about. That is, a basis of quark states can be labeled by  $|flavor, color, spin\rangle$ . Here the *flavor* label runs over the values  $u, d, s$  and provides a representation of the **3** of  $SU(3)_{\text{flavor}}$ . The *color* label runs over the values  $r, g, b$  and provides a representation of the **3** of  $SU(3)_{\text{color}}$ . Finally the *spin* label runs over the values  $\uparrow, \downarrow$  and provides a representation of the **2** of the  $SU(2)$  angular momentum group. One sometimes says that quarks are in the **(3, 3, 2)** representation of the  $SU(3)_{\text{flavor}} \times SU(3)_{\text{color}} \times SU(2)_{\text{spin}}$  symmetry group.†

Strangely enough, color has never been observed directly in the lab. What I mean by this is that (for example) hadrons can be grouped into multiplets that are in non-trivial representations of  $SU(3)_{\text{flavor}}$ . But there are no degeneracies in the hadron spectrum associated with  $SU(3)_{\text{color}}$ : all observed particles are color singlets. We’ll elevate this observation to the status of a principle, and postulate that *all hadrons are invariant under  $SU(3)_{\text{color}}$  transformations*. This implies quark confinement: since quarks are in the **3** of  $SU(3)_{\text{color}}$  they can’t appear in isolation. What’s nice is that we *can* make color-singlet mesons and baryons. Denoting quark color by a 3-component vector  $z^a$  we can make

$$\begin{array}{ll} \text{color-singlet baryon wavefunctions} & \epsilon^{abc} \\ \text{color-singlet meson wavefunctions} & \delta^a_b \end{array}$$

So why introduce color at all? It provides a way to restore Fermi statistics. For example, for the  $\Delta^{++}$  baryon, the color wavefunction is totally antisymmetric. So when we combine it with the totally symmetric flavor and spin wavefunction given above we get a state that respects Fermi statistics.

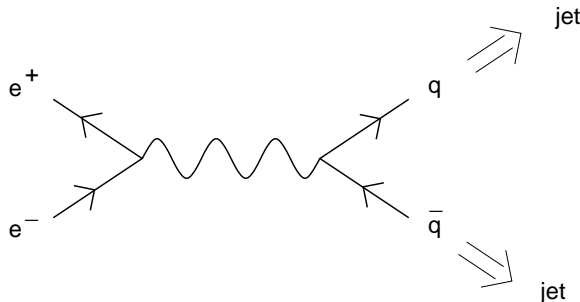
† It gets confusing, but try to keep in mind that  $SU(3)_{\text{flavor}}$  and  $SU(3)_{\text{color}}$  are completely separate symmetries that have nothing to do with one another.

### 3.2 Evidence for quarks

This may be starting to seem very contrived. But in fact there's very concrete evidence that quarks carry the spin, charge and color quantum numbers we've assigned.

#### 3.2.1 Quark spin

Perhaps the most direct evidence that quarks carry spin 1/2 comes from the process  $e^+e^- \rightarrow \text{two jets}$ . This can be viewed as a two-step process: an electromagnetic interaction  $e^+e^- \rightarrow q\bar{q}$ , followed by strong interactions which convert the  $q$  and  $\bar{q}$  into jets of (color-singlet) hadrons.



Assuming the quark and antiquark don't interact significantly in the final state, each jet carries the full momentum of its parent quark or antiquark. Thus by measuring the angular distribution of jets you can directly determine the angular distribution of  $q\bar{q}$  pairs produced in the process  $e^+e^- \rightarrow q\bar{q}$ . For spin-1/2 quarks this is governed by the differential cross section<sup>†</sup>

$$\frac{d\sigma}{d\Omega} = \frac{Q_e^2 Q_q^2 e^4}{64\pi^2 s} (1 + \cos^2 \theta) . \quad (3.1)$$

Here we're working in the center of mass frame and neglecting the electron and quark masses.  $Q_e$  is the electron charge and  $Q_q$  is the quark charge, both measured in units of  $e \equiv \sqrt{4\pi\alpha}$ , while  $s = (p_1 + p_2)^2$  is the square of the total center-of-mass energy and  $\theta$  is the c.m. scattering angle (measured with respect to the beam direction).

As you'll show in problem 4.1, this angular distribution is characteristic of having spin-1/2 particles in the final state. The data indicates that quarks indeed carry spin 1/2: Hanson et. al., *Phys. Rev. Lett.* **35** (1975) 1609.

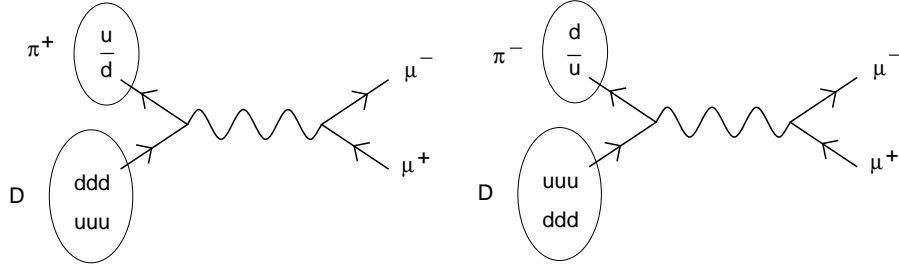
<sup>†</sup> For example see Peskin & Schroeder section 5.1. We'll discuss this in detail in the next chapter.

### 3.2.2 Quark charge

One can measure (ratios of) quark charges using the so-called Drell-Yan process

$$\pi^\pm \text{ deuteron} \rightarrow \mu^+ \mu^- \text{ anything}.$$

Recall that  $\pi^+ \sim u\bar{d}$  and  $\pi^- \sim d\bar{u}$ , while the deuteron (if you think of it as a proton plus neutron) has quark content  $uuuddd$ . We can regard the Drell-Yan process as an elementary electromagnetic interaction  $q\bar{q} \rightarrow \mu^+\mu^-$  together with lots of strong interactions. In cartoon form the interactions are



At high energies the electromagnetic process has a center-of-mass cross section

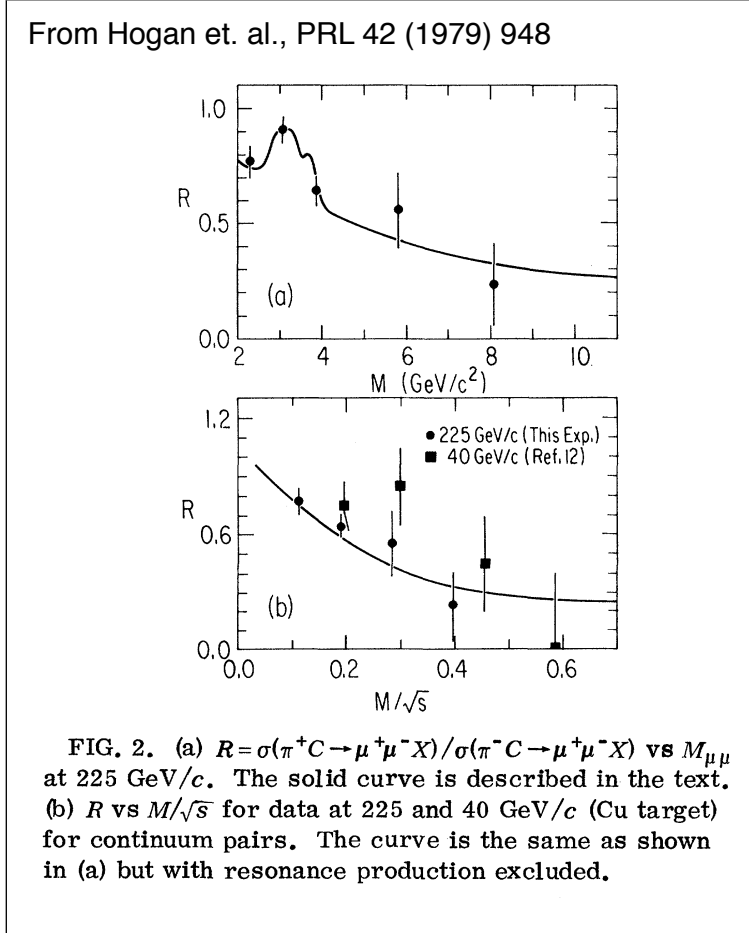
$$\sigma = \frac{Q_q^2 Q_\mu^2 e^4}{12\pi s}$$

that follows from integrating (3.1) over angles. You might worry that the whole process is dominated by strong interactions. What saves us is the fact that the deuteron is an isospin singlet.<sup>†</sup> This means that – since isospin is a symmetry of the strong interactions – strong interactions can't distinguish between the initial states  $\pi^+ D$  and  $\pi^- D$ . They only contribute an overall factor to the two cross sections, which cancels out when we take the ratio. Thus we can predict

$$\frac{\sigma(\pi^+ D \rightarrow \mu^+ \mu^- X)}{\sigma(\pi^- D \rightarrow \mu^+ \mu^- X)} \approx \frac{Q_d^2}{Q_u^2} = \frac{(-1/3)^2}{(2/3)^2} = \frac{1}{4}.$$

This fits the data (actually taken with an isoscalar  $^{12}\text{C}$  target) pretty well. See Hogan et. al., *Phys. Rev. Lett.* **42** (1979) 948.

<sup>†</sup> It's a proton-neutron bound state with no orbital angular momentum, isospin  $I = 0$ , and regular spin  $J = 1$ .



### 3.2.3 Quark color

A particularly elegant piece of evidence for quark color comes from the decay  $\pi^0 \rightarrow \gamma\gamma$ , as you'll see in problem 13.1. But for now a nice quantity to study is the cross-section ratio

$$R = \frac{\sigma(e^+e^- \rightarrow \text{hadrons})}{\sigma(e^+e^- \rightarrow \mu^+\mu^-)}.$$

The initial step in the reaction  $e^+e^- \rightarrow \text{hadrons}$  is the purely electrodynamic process  $e^+e^- \rightarrow q\bar{q}$ , followed by strong interactions that turn the  $q$  and  $\bar{q}$  into a collection of hadrons. This “hadronization” takes place with unit probability, so we don’t need to worry about it, and we have

$$R = \sum_{\text{quarks}} \left| \begin{array}{c} e^+ \\ | \\ e^- \end{array} \right\rangle \left[ \text{diagram} \right] \left| \begin{array}{c} q \\ | \\ \bar{q} \end{array} \right\rangle^2 / \left| \begin{array}{c} e^+ \\ | \\ e^- \end{array} \right\rangle \left[ \text{diagram} \right] \left| \begin{array}{c} \mu^- \\ | \\ \mu^+ \end{array} \right\rangle^2$$

$$R = \sum_{\text{quarks}} Q_{\text{quark}}^2$$
$$R = 3 \left[ \underbrace{(2/3)^2}_{\text{up}} + \underbrace{(-1/3)^2}_{\text{down}} + \underbrace{(-1/3)^2}_{\text{strange}} \right] = 2$$

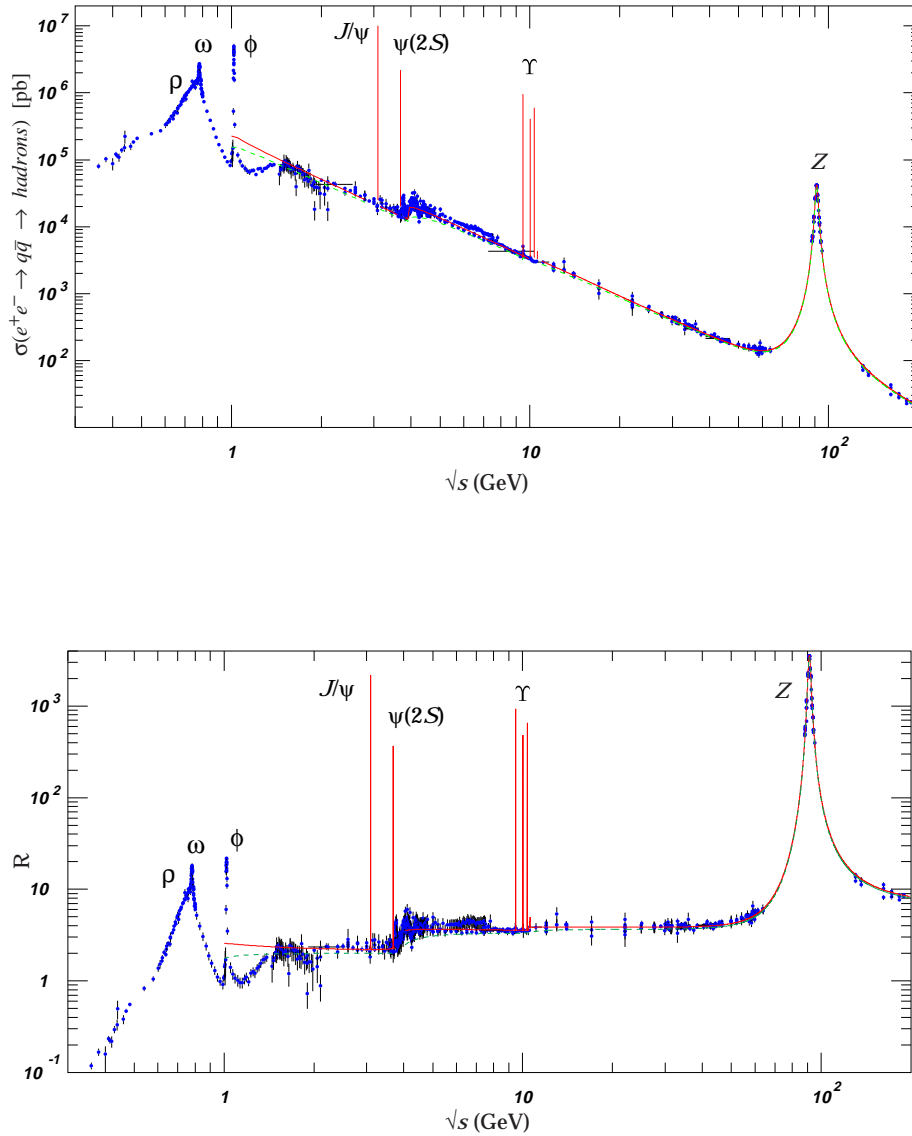
charm	$m_c = 1.3 \text{ GeV}$	$Q_c = 2/3$
bottom	$m_b = 4.2 \text{ GeV}$	$Q_b = -1/3$
top	$m_t = 172 \text{ GeV}$	$Q_t = 2/3$

$$R = 3 \left[ (2/3)^2 + (-1/3)^2 + (-1/3)^2 + (2/3)^2 + (-1/3)^2 \right] = 11/3$$

## References

Evidence for the existence of quarks is given in chapter 1 of Quigg, under the heading “why we believe in quarks.”

## 010001-6 39. Plots of cross sections and related quantities

 $\sigma$  and  $R$  in  $e^+e^-$  Collisions

**Figure 39.6, Figure 39.7:** World data on the total cross section of  $e^+e^- \rightarrow \text{hadrons}$  and the ratio  $R = \sigma(e^+e^- \rightarrow \text{hadrons}) / \sigma(e^+e^- \rightarrow \mu^+\mu^-)$ , QED simple pole). The curves are an educative guide. The solid curves are the 3-loop pQCD predictions for  $\sigma(e^+e^- \rightarrow \text{hadrons})$  and the  $R$  ratio, respectively [see our Review on Quantum chromodynamics, Eq. (9.12)] or, for more details, K.G. Chetyrkin *et al.*, Nucl. Phys. **B586**, 56 (2000), Eqs. (1)–(3)). Breit-Wigner parameterizations of  $J/\psi$ ,  $\psi(2S)$ , and  $\Upsilon(nS)$ ,  $n = 1..4$  are also shown. **Note:** The experimental shapes of these resonances are dominated by the machine energy spread and are not shown. The dashed curves are the naive quark parton model predictions for  $\sigma$  and  $R$ . The full list of references, as well as the details of  $R$  ratio extraction from the original data, can be found in O.V. Zenin *et al.*, hep-ph/0110176 (to be published in J. Phys. **G**). Corresponding computer-readable data files are available at [http://wwwppds.ihep.su/~zenin\\_o/contents\\_plots.html](http://wwwppds.ihep.su/~zenin_o/contents_plots.html). (Courtesy of the COMPAS (Protvino) and HEPDATA (Durham) Groups, November 2001.)

**Exercises****3.1 Decays of the  $W$  and  $\tau$** 

The  $W^-$  boson decays to a “weak doublet” pair of fermions, meaning either  $e^- \bar{\nu}_e$ ,  $\mu^- \bar{\nu}_\mu$ ,  $\tau^- \bar{\nu}_\tau$ ,  $\bar{u}d$ ,  $\bar{c}s$ , or in principle  $\bar{t}b$ .

- (i) Suppose the amplitude for  $W^-$  decay is the same for all fermion pairs. Only kinematically allowed decays are possible, but aside from that you can neglect differences in phase space due to fermion masses. Predict the branching ratios for the decays

$$W^- \rightarrow e^- \bar{\nu}_e$$

$$W^- \rightarrow \mu^- \bar{\nu}_\mu$$

$$W^- \rightarrow \tau^- \bar{\nu}_\tau$$

$$W^- \rightarrow \text{hadrons}$$

- (ii) The  $\tau^-$  lepton decays by  $\tau^- \rightarrow W^- \nu_\tau$ , followed by  $W^-$  decay. Predict the branching ratios for the decays

$$\tau^- \rightarrow \nu_\tau e^- \bar{\nu}_e$$

$$\tau^- \rightarrow \nu_\tau \mu^- \bar{\nu}_\mu$$

$$\tau^- \rightarrow \nu_\tau \text{ hadrons}$$

- (iii) How did you do, compared to the particle data book? What would happen if you didn't take color into account?



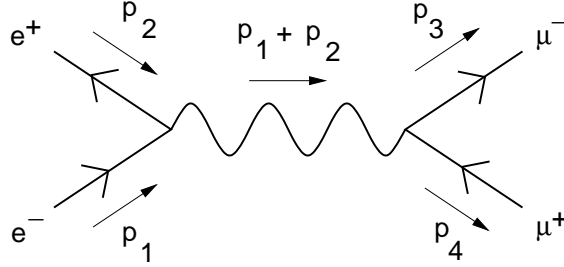
## 4

### Chiral spinors and helicity amplitudes

Physics 85200

January 6, 2014

To be concrete let me focus on the process  $e^+e^- \rightarrow \mu^+\mu^-$ .



Evaluating this diagram is a straightforward exercise in Feynmanology, as reviewed in appendix A. The amplitude is

$$-i\mathcal{M} = \bar{v}(p_2)(-ieQ\gamma^\mu)u(p_1)\frac{-ig_{\mu\nu}}{(p_1 + p_2)^2}\bar{u}(p_3)(-ieQ\gamma^\nu)v(p_4)$$

where  $Q = -1$  for the electron and muon. In the center of mass frame the corresponding differential cross section is

$$\frac{d\sigma}{d\Omega} = \frac{e^4}{64\pi^2 s} \sqrt{\frac{s - 4m_\mu^2}{s - 4m_e^2}} \left( 1 + \left(1 - \frac{4m_e^2}{s}\right)\left(1 - \frac{4m_\mu^2}{s}\right)\cos^2\theta + \frac{4(m_e^2 + m_\mu^2)}{s} \right).$$

Here  $s = (p_1 + p_2)^2$  is the square of the total center-of-mass energy and  $\theta$  is the c.m. scattering angle (measured with respect to the beam direction). This result is clearly something of a mess, however note that things simplify quite a bit in the high-energy (or equivalently massless) limit  $\sqrt{s} \gg m_e, m_\mu$ . In this limit we have

$$\frac{d\sigma}{d\Omega} = \frac{e^4}{64\pi^2 s} (1 + \cos^2\theta).$$

One of our main goals in this section is to understand the origin of this simplification.

### 4.1 Chiral spinors

I'll start with some facts about Dirac spinors. Recall that a Dirac spinor  $\psi_D$  is a 4-component object. Under a Lorentz transform

$$\psi_D \rightarrow e^{-i(\vec{\theta} \cdot \mathbf{J} + \vec{\phi} \cdot \mathbf{K})} \psi_D. \quad (4.1)$$

Here we're performing a rotation through an angle  $|\vec{\theta}|$  about the direction  $\hat{\theta}$ , and we're boosting with rapidity  $|\vec{\phi}|$  in the direction  $\hat{\phi}$ . The rotation generators  $\mathbf{J}$  and boost generators  $\mathbf{K}$  are given in terms of Pauli matrices by

$$\mathbf{J} = \begin{pmatrix} \vec{\sigma}/2 & 0 \\ 0 & \vec{\sigma}/2 \end{pmatrix} \quad \mathbf{K} = \begin{pmatrix} -i\vec{\sigma}/2 & 0 \\ 0 & i\vec{\sigma}/2 \end{pmatrix}$$

Here I'm working in the “chiral basis” where the Dirac matrices take the form

$$\gamma^0 = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix} \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$$

What's nice about the chiral basis is that the Lorentz generators are block-diagonal. This makes it manifest that Dirac spinors are a *reducible* representation of the Lorentz group. The irreducible pieces of  $\psi_D$  are obtained by decomposing

$$\psi_D = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$$

into left- and right-handed “chiral spinors”  $\psi_L, \psi_R$ .

In the chiral basis

$$\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} -\mathbf{1} & 0 \\ 0 & \mathbf{1} \end{pmatrix}.$$

We can use this to define projection operators

$$P_L = \frac{1 - \gamma^5}{2} = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & 0 \end{pmatrix} \quad P_R = \frac{1 + \gamma^5}{2} = \begin{pmatrix} 0 & 0 \\ 0 & \mathbf{1} \end{pmatrix}$$

which pick out the left- and right-handed pieces of a Dirac spinor.

$$P_L \psi_D = \begin{pmatrix} \psi_L \\ 0 \end{pmatrix} \quad P_R \psi_D = \begin{pmatrix} 0 \\ \psi_R \end{pmatrix}.$$

To see why this decomposition is useful, let's express the QED Lagrangian in terms of chiral spinors.

$$\mathcal{L}_{\text{QED}} = \bar{\psi} (i\gamma^\mu D_\mu - m) \psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

Here  $D_\mu = \partial_\mu + ieQA_\mu$  is the covariant derivative. In the chiral basis we have

$$\begin{aligned} & \bar{\psi} (i\gamma^\mu D_\mu - m) \psi \\ &= \begin{pmatrix} \psi_L^\dagger & \psi_R^\dagger \end{pmatrix} \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix} \begin{pmatrix} -m & iD_0 + i\vec{D} \cdot \vec{\sigma} \\ iD_0 - i\vec{D} \cdot \vec{\sigma} & -m \end{pmatrix} \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} \\ &= \begin{pmatrix} \psi_L^\dagger & \psi_R^\dagger \end{pmatrix} \begin{pmatrix} iD_0 - i\vec{D} \cdot \vec{\sigma} & -m \\ -m & iD_0 + i\vec{D} \cdot \vec{\sigma} \end{pmatrix} \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} \\ &= i\psi_L^\dagger (D_0 - \vec{D} \cdot \vec{\sigma}) \psi_L + i\psi_R^\dagger (D_0 + \vec{D} \cdot \vec{\sigma}) \psi_R - m (\psi_L^\dagger \psi_R + \psi_R^\dagger \psi_L) \end{aligned}$$

The important thing to note is that the mass term couples  $\psi_L$  to  $\psi_R$ . But in the massless limit  $\psi_L$  and  $\psi_R$  behave as two independent fields. They're both coupled to the electromagnetic field, of course, through the interaction Hamiltonian

$$\mathcal{H}_{\text{int}} = -\mathcal{L}_{\text{int}} = eQ \left[ \psi_L^\dagger (A_0 - \mathbf{A} \cdot \vec{\sigma}) \psi_L + \psi_R^\dagger (A_0 + \mathbf{A} \cdot \vec{\sigma}) \psi_R \right]. \quad (4.2)$$

Note that there are no  $\psi_L \psi_R A$  couplings in the Hamiltonian. This will lead to simplifications in high-energy scattering amplitudes.

To see the physical interpretation of these chiral spinors recall the plane wave solutions to the Dirac equation worked out in Peskin & Schroeder. We're interested in describing states with definite

helicity  $\equiv$  component of spin along direction of motion.

To describe these states let  $\hat{p}$  be a unit vector in the direction of motion. Start by finding the (orthonormal) eigenvectors of the operator  $\hat{p} \cdot \vec{\sigma}$ :

$$(\hat{p} \cdot \vec{\sigma}) \xi^\pm = \pm \xi^\pm \quad |\xi^+|^2 = |\xi^-|^2 = 1.$$

Then you can construct Dirac spinors describing states with definite helicity.<sup>†</sup>

<sup>†</sup> It may seem counterintuitive that  $v_R$  is constructed from  $\xi^-$ , and  $v_L$  from  $\xi^+$ . To understand this you can either go through some intellectual contortions with hole theory, or more straightforwardly you can read it off from the angular momentum operator of a quantized Dirac field.

$$\begin{aligned}
u_R(p) &= \begin{pmatrix} \sqrt{E-|\mathbf{p}|} \xi^+ \\ \sqrt{E+|\mathbf{p}|} \xi^+ \end{pmatrix} && \text{right-handed particle (helicity } +\hbar/2) \\
u_L(p) &= \begin{pmatrix} \sqrt{E+|\mathbf{p}|} \xi^- \\ \sqrt{E-|\mathbf{p}|} \xi^- \end{pmatrix} && \text{left-handed particle (helicity } -\hbar/2) \\
v_R(p) &= \begin{pmatrix} \sqrt{E+|\mathbf{p}|} \xi^- \\ -\sqrt{E-|\mathbf{p}|} \xi^- \end{pmatrix} && \text{right-handed antiparticle (helicity } +\hbar/2) \\
v_L(p) &= \begin{pmatrix} \sqrt{E-|\mathbf{p}|} \xi^+ \\ -\sqrt{E+|\mathbf{p}|} \xi^+ \end{pmatrix} && \text{left-handed antiparticle (helicity } -\hbar/2)
\end{aligned}$$

These spinors are kind of messy. But in the massless limit  $E \rightarrow |\mathbf{p}|$  and things simplify a lot:

$$\begin{aligned}
u_R(p) &\rightarrow \begin{pmatrix} 0 \\ \sqrt{2E} \xi^+ \end{pmatrix} && \text{pure } \psi_R \\
u_L(p) &\rightarrow \begin{pmatrix} \sqrt{2E} \xi^- \\ 0 \end{pmatrix} && \text{pure } \psi_L \\
v_R(p) &\rightarrow \begin{pmatrix} \sqrt{2E} \xi^- \\ 0 \end{pmatrix} && \text{pure } \psi_L \\
v_L(p) &\rightarrow \begin{pmatrix} 0 \\ -\sqrt{2E} \xi^+ \end{pmatrix} && \text{pure } \psi_R
\end{aligned}$$

Thus in the massless limit

$\psi_L$  describes a left-handed particle and its right-handed antiparticle  
 $\psi_R$  describes a right-handed particle and its left-handed antiparticle

WARNING: when people talk about left- or right-handed *particles* they're referring to helicity  $\equiv$  component of spin along the direction of motion. When people talk about left- or right-handed *spinors* they're referring to chirality  $\equiv$  behavior under Lorentz transforms. In general these are very different notions although, as we've seen, they get tied together in the massless limit.

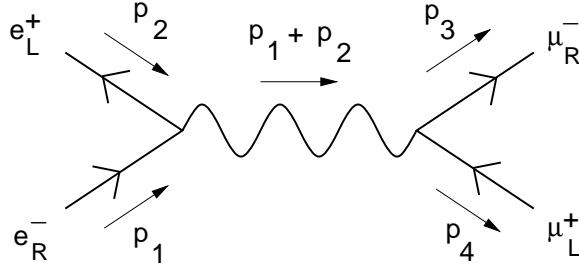
## 4.2 Helicity amplitudes

Let's look more closely at the high-energy behavior of  $e^+e^- \rightarrow \mu^+\mu^-$ . At high energies the electron and muon masses can be neglected, which makes it useful to work in terms of chiral spinors. The interaction Hamiltonian looks like two copies of (4.2), one for the electron and one for the muon.  $\mathcal{H}_{\text{int}}$  only couples two spinors of the same chirality to the gauge field, so out of the 16 possible scattering amplitudes between states of definite helicity only four are non-zero:

$$e_L^+e_R^- \rightarrow \mu_L^+\mu_R^- \quad e_L^+e_R^- \rightarrow \mu_R^+\mu_L^- \quad e_R^+e_L^- \rightarrow \mu_L^+\mu_R^- \quad e_R^+e_L^- \rightarrow \mu_R^+\mu_L^-$$

Here I'm denoting the helicity of the particles with subscripts  $L, R$ . For example, both  $e_L^+$  and  $e_R^-$  sit inside a right-handed spinor. Ditto for  $\mu_L^+$  and  $\mu_R^-$ . In general this is known as “helicity conservation at high energies” (see Halzen & Martin section 6.6).

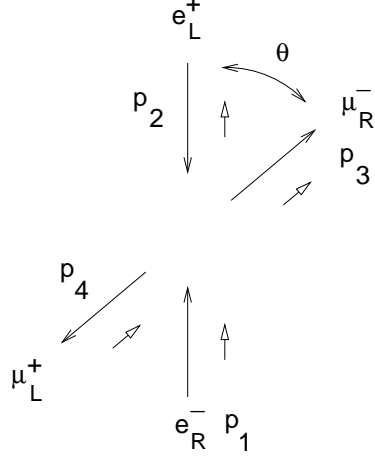
Let's study the particular spin-polarized process  $e_L^+e_R^- \rightarrow \mu_L^+\mu_R^-$ .



The amplitude is

$$-i\mathcal{M} = \bar{v}_L(p_2)(-ieQ\gamma^\mu)u_R(p_1)\frac{-ig_{\mu\nu}}{(p_1 + p_2)^2}\bar{u}_R(p_3)(-ieQ\gamma^\nu)v_L(p_4).$$

At this point it's convenient to fix the kinematics (spatial momenta indicated by large arrows, spins indicated by small arrows)



Then for the incoming  $e^+e^-$  we have (I'm only interested in the angular dependence, so I'm not going to worry about normalizing the spinors)

$$p_1 = (E, 0, 0, E) \quad p_2 = (E, 0, 0, -E)$$

$$u_R(p_1) = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad v_L(p_2) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Then the 'electron current' part of the diagram is

$$\begin{aligned} & \bar{v}_L(p_2)(-ieQ\gamma^\mu)u_R(p_1) \\ & \sim v_L^\dagger(p_2) \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix} \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}; \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} u_R(p_1) \\ & = v_L^\dagger(p_2) \begin{pmatrix} \mathbf{1} & 0 \\ 0 & \mathbf{1} \end{pmatrix}; \begin{pmatrix} -\sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix} u_R(p_1) \\ & = (0, 1, i, 0) \end{aligned} \tag{4.3}$$

To get the 'muon current' part of the diagram, first consider scattering at  $\theta = 0$ , for which

$$p_3 = (E, 0, 0, E) \quad p_4 = (E, 0, 0, -E)$$

$$u_R(p_3) = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad v_L(p_4) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\Rightarrow \quad \bar{u}_R(p_3)(-ieQ\gamma^\mu)v_L(p_4)$$

$$\begin{aligned}
& \sim u_R^\dagger(p_3) \left( \begin{pmatrix} \mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix}; \begin{pmatrix} -\sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix} \right) v_L(p_4) \\
& = (0, 1, -i, 0)
\end{aligned}$$

To get the result for general  $\theta$  we just need to rotate this 4-vector through an angle  $\theta$  about (say) the  $y$ -axis:

$$\bar{u}_R(p_3)(-ieQ\gamma^\mu)v_L(p_4) \sim (0, \cos\theta, -i, \sin\theta).$$

The helicity amplitude goes like the dot product of the two currents:

$$\mathcal{M}(e_L^+ e_R^- \rightarrow \mu_L^+ \mu_R^-) \sim (0, 1, i, 0) \cdot (0, \cos\theta, -i, \sin\theta) = -(1 + \cos\theta).$$

This result is a beautifully simple example of quantum measurement at work. The electron current describes an initial state with one unit of angular momentum polarized in the  $+z$  direction  $|J = 1, J_z = 1\rangle$ . To verify this statement, just look at how the 4-vector (4.3) transforms under a rotation about the  $z$  axis. The (complex conjugate of the) muon current describes a final state which also has one unit of angular momentum, but polarized in the direction of the outgoing muon:  $|J = 1, J_{\mu^-} = 1\rangle$ . The angular dependence of the amplitude is given by the inner product of these two angular momentum eigenstates.<sup>†</sup> As a reality check, note that the amplitude vanishes when  $\theta = \pi$  (the amplitude for an eigenstate with  $J_z = +1$  to be found in a state with  $J_z = -1$  vanishes).

The other helicity amplitudes go through in pretty much the same way. The only difference is that for a process like  $LR \rightarrow RL$  it's scattering at  $\theta = 0$  that's prohibited; this shows up as a  $(1 - \cos\theta)$  dependence. Finally, the cross sections go like  $|\mathcal{M}|^2$ , so

$$\begin{aligned}
\left(\frac{d\sigma}{d\Omega}\right)_{LR \rightarrow LR} &= \left(\frac{d\sigma}{d\Omega}\right)_{RL \rightarrow RL} \sim (1 + \cos\theta)^2 \\
\left(\frac{d\sigma}{d\Omega}\right)_{LR \rightarrow RL} &= \left(\frac{d\sigma}{d\Omega}\right)_{RL \rightarrow LR} \sim (1 - \cos\theta)^2
\end{aligned}$$

Summing over final polarizations and averaging over initial polarizations gives

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{unpolarized}} \sim 1 + \cos^2\theta.$$

Although spin-averaged amplitudes are usually easier to compute, especially for finite fermion masses, it's often easier to interpret helicity amplitudes.

<sup>†</sup> This sort of analysis is quite general. See appendix B.

## References

Plane wave solutions to the Dirac equation are worked out in Peskin & Schroeder: for the classical theory see p. 48, for the (slightly confusing) quantum interpretation see p. 61, for a summary of the results see appendix A.2. Chiral spinors (also known as Weyl spinors) are discussed in section 3.2 of Peskin & Schroeder, while helicity amplitudes are covered in section 5.2.

## Exercises

### 4.1 Quark spin and jet production

Two-jet production in  $e^+e^-$  collisions can be understood as a tree-level QED-like process  $e^+e^- \rightarrow \gamma \rightarrow q\bar{q}$  followed by hadronization of the quark and antiquark. Assuming the quark and antiquark don't interact significantly, each jet carries the full momentum of its parent quark or antiquark. The distribution of jets with respect to the scattering angle  $\theta$  carries information about the spin of a quark. References: there's some discussion in Cheng & Li p. 215-216, and for a nice picture see p. 9 in Quigg.

- (i) Suppose the quark is a spin-1/2 Dirac fermion with charge  $Q$  and mass  $M$ . What is the center of mass differential cross section for the process  $e^+e^- \rightarrow q\bar{q}$ ? You should average over initial spins and sum over final spins, also you should keep track of the dependence on both the electron and quark masses.
- (ii) Now suppose the quark is a spinless particle that can be modeled as a complex scalar field with charge  $Q$  and mass  $M$ . Re-evaluate the center of mass differential cross-section for  $e^+e^- \rightarrow q\bar{q}$ . You should average over the initial  $e^+e^-$  spins. The Feynman rules are in appendix A.
- (iii) In the high-energy limit the electron and quark masses are negligible and the angular distribution simplifies. For spin-1/2 quarks there's a nice explanation for the angular distribution at high energies: we talked about it in class, or see Peskin & Schroeder sect. 5.2 or Halzen & Martin sect. 6.6. What's the analogous explanation for the high energy angular distribution of spinless quarks?
- (iv) In  $e^+e^-$  collisions at  $E_{\text{cm}} = 7.4$  GeV the jet-axis angular distribution was found to be proportional to  $1 + (0.78 \pm 0.12) \cos^2 \theta$  [Phys. Rev. Lett. 35, 1609 (1975)]. What's the spin of a quark?



## 5

### Spontaneous symmetry breaking

Physics 85200

January 6, 2014

#### 5.1 Symmetries and conservation laws

When discussing symmetries it's convenient to use the language of Lagrangian mechanics. The prototype example I'll have in mind is a scalar field  $\phi(t, \mathbf{x})$  with potential energy  $V(\phi)$ . The action is

$$S[\phi] = \int d^4x \mathcal{L}(\phi, \partial\phi) \quad \mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - V(\phi)$$

Classical trajectories correspond to stationary points of the action.

$$\begin{aligned} \text{vary } \phi \rightarrow \phi + \delta\phi \\ \delta S = 0 \text{ to first order in } \delta\phi \end{aligned} \quad \Leftrightarrow \quad \phi \text{ is a classical trajectory}$$

With suitable boundary conditions on  $\delta\phi$  this variational principle is equivalent to the Euler-Lagrange equations

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} - \frac{\partial \mathcal{L}}{\partial \phi} = 0.$$

To see this one computes

$$\begin{aligned} \delta S &= \int d^4x \left( \frac{\partial \mathcal{L}}{\partial \phi} \delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta\partial_\mu \phi \right) \\ &= \int d^4x \left( \frac{\partial \mathcal{L}}{\partial \phi} \delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\mu \delta\phi \right) \\ &= \int d^4x \delta\phi \left( \frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) + \text{surface terms} \end{aligned}$$

With suitable boundary conditions we can drop the surface terms, in which case  $\delta S$  vanishes for any  $\delta\phi$  iff the Euler-Lagrange equations are satisfied.

Now let's discuss *continuous internal symmetries*, which are transformations of the fields that

- depend on one or more continuous parameters,
- are “internal,” in the sense that the transformation can depend on  $\phi$  but not on  $\partial_\mu\phi$ ,
- are “symmetries,” in the sense that they leave the Lagrangian invariant.

That is, we consider continuous transformations of the fields

$$\phi(x) \rightarrow \phi'(x) = \phi'(\phi(x)) \quad (5.1)$$

such that

$$\mathcal{L}(\phi, \partial\phi) = \mathcal{L}(\phi', \partial\phi').$$

The notation in (5.1) means that the new value of the field at the point  $x$  only depends on the old value of the field at the point  $x$  – it doesn't depend on the old value of the field at any other point. Equivalently the transformation can depend on  $\phi$  but not on  $\partial_\mu\phi$ .

One consequence of this definition is that if  $\phi(x)$  satisfies the equations of motion, then so does  $\phi'(x)$ . In other words a symmetry maps one solution to the equations of motion into another solution.<sup>†</sup>

Another consequence is Noether's theorem, that symmetries imply conservation laws. Given an infinitesimal internal symmetry transformation  $\phi \rightarrow \phi + \delta\phi$  the current

$$j^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi \quad (5.2)$$

is conserved. That is, the equations of motion for  $\phi$  imply that  $\partial_\mu j^\mu = 0$ .

Proof: Suppose  $\phi$  satisfies the equations of motion, and consider an infinitesimal internal symmetry transformation  $\phi \rightarrow \phi + \delta\phi$ . Let's compute  $\delta\mathcal{L}$  in two ways. On the one hand  $\delta\mathcal{L} = 0$  by the definition of an internal symmetry. On the other hand

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\phi}\delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta\partial_\mu\phi.$$

<sup>†</sup> Proof: the Lagrangian is invariant so  $S[\phi] = S[\phi']$ . Varying with respect to  $\phi(x)$  the chain rule gives  $\frac{\delta S}{\delta\phi(x)} = \int_y \frac{\delta S}{\delta\phi'(y)} \frac{\delta\phi'(y)}{\delta\phi(x)}$ . Assuming the Jacobian  $\frac{\delta\phi'}{\delta\phi}$  is non-singular we have  $\frac{\delta S}{\delta\phi} = 0$  iff  $\frac{\delta S}{\delta\phi'} = 0$ .

If we use the Euler-Lagrange equations this becomes

$$\begin{aligned}\delta\mathcal{L} &= \partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta\partial_\mu\phi \\ &= \partial_\mu \left( \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi \right)\end{aligned}$$

Comparing the two expressions for  $\delta\mathcal{L}$  we see that  $j^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi$  is conserved: the equations of motion for  $\phi$  imply  $\partial_\mu j^\mu = 0$ . Q.E.D.

This shows that symmetries  $\Rightarrow$  conservation laws. It works the other way as well: given a conservation law obtained using Noether's theorem, you can reconstruct the symmetry it came from. This is best expressed in Hamiltonian language. Given a conserved current  $j^\mu$  one has the corresponding conserved charge

$$Q = \int d^3x j^0(t, \mathbf{x}) \quad \left( \frac{dQ}{dt} = 0 \text{ so } t \text{ is arbitrary} \right)$$

$Q$  is the *generator* of the symmetry in the sense that

$$i [Q, \phi(t, \mathbf{x})] = \delta\phi(t, \mathbf{x}).$$

(This is a quantum commutator, but if you like Poisson brackets you can write a corresponding expression in the classical theory.)

Proof: recall that the canonical momentum

$$\pi = \frac{\partial\mathcal{L}}{\partial(\partial_0\phi)}$$

obeys the equal-time commutator

$$i [\pi(t, \mathbf{x}), \phi(t, \mathbf{y})] = \delta^3(\mathbf{x} - \mathbf{y})$$

and note that  $Q = \int d^3x j^0 = \int d^3x \pi \delta\phi$  satisfies

$$\begin{aligned}i [Q, \phi(t, \mathbf{x})] &= i \int d^3y [\pi(t, \mathbf{y}) \delta\phi(t, \mathbf{y}), \phi(t, \mathbf{x})] \\ &= i \int d^3y [\pi(t, \mathbf{y}), \phi(t, \mathbf{x})] \delta\phi(t, \mathbf{y}) \\ &= \delta\phi(t, \mathbf{x}).\end{aligned}$$

It's important for this argument that  $\delta\phi(t, \mathbf{x})$  commutes with  $\phi(t, \mathbf{x})$ . This is valid for internal symmetries, since  $\delta\phi(x)$  only depends on  $\phi(x)$  and  $\phi(x)$  commutes with itself.

### 5.1.1 Flavor symmetries of the quark model

As an example, consider the quark model we introduced in section 3.1. In terms of a collection of Dirac spinor fields

$$\psi = \begin{pmatrix} u \\ d \\ s \end{pmatrix}$$

we guessed that the strong interaction Lagrangian looked like

$$\mathcal{L}_{\text{strong}} = \bar{\psi} i \gamma^\mu \partial_\mu \psi + \dots$$

The quark kinetic terms are invariant under  $U(3)$  transformations  $\psi \rightarrow U\psi$ . If we assume this symmetry extends to all of  $\mathcal{L}_{\text{strong}}$ , we can derive the corresponding conserved currents. Setting  $U = e^{-i\lambda^a T^a}$  where the generators  $T^a$  are a basis of  $3 \times 3$  Hermitian matrices we have<sup>†</sup>

$$\text{infinitesimal transformation } \delta\psi = -i\lambda^a T^a \psi$$

$$\frac{\partial^R \mathcal{L}_{\text{strong}}}{\partial(\partial_\mu \psi)^R} = \bar{\psi} i \gamma^\mu \quad \frac{\partial^R \mathcal{L}_{\text{strong}}}{\partial(\partial_\mu \bar{\psi})^R} = 0$$

$$\Rightarrow j^{\mu a} = \bar{\psi} \gamma^\mu T^a \psi \quad \text{conserved}$$

Here I'm using the fermionic version of Noether's theorem, worked out in problem 5.1. In the last line I stripped off the infinitesimal parameters  $\lambda^a$ . As promised, this approach unifies several conservation laws introduced in chapter 1.

$U(3)$ generator	conservation law
$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	quark number ( = $B/3$ )
$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$	strangeness
$\begin{pmatrix} \sigma^i & 0 \\ 0 & 0 \end{pmatrix}$	isospin
traceless	flavor $SU(3)$

<sup>†</sup> There is a subtle point here - what if there were terms involving  $\partial_\mu \psi$  or  $\partial_\mu \bar{\psi}$  hidden in the  $\dots$  terms in  $\mathcal{L}_{\text{strong}}$ ? Such terms would modify the currents, but fortunately dimensional analysis tells you that any such terms can be neglected at low energies.

## 5.2 Spontaneous symmetry breaking (classical)

It might seem that we've exhausted our discussion of symmetries and their consequences. But there's a somewhat surprising phenomenon that can occur in quantum field theory: the ground state of a quantum field doesn't have to be unique. This opens up a new possibility: given some degenerate vacua, a symmetry transformation  $\phi \rightarrow \phi'$  can leave the Lagrangian invariant but may act non-trivially on the space of vacua.

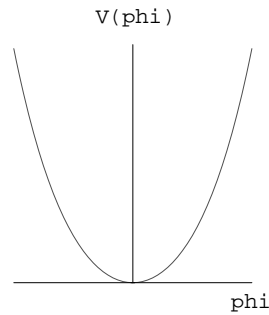
This phenomenon is known as spontaneous symmetry breaking. Rather than give a general discussion, I'll go through a few examples that illustrate how it works. In this section the analysis will be mostly classical. In the next section we'll explore the consequences of spontaneous symmetry breaking in the quantum theory.

### 5.2.1 Breaking a discrete symmetry

As a first example, consider a real scalar field with a  $\phi^4$  self-interaction.

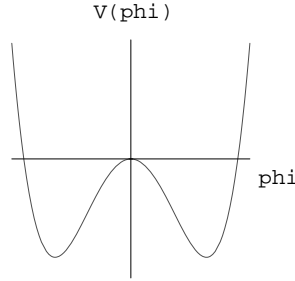
$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} \mu^2 \phi^2 - \frac{1}{4} \lambda \phi^4$$

The potential energy of the field  $V(\phi) = \frac{1}{2} \mu^2 \phi^2 + \frac{1}{4} \lambda \phi^4$ . Assuming  $\mu^2 > 0$  the potential looks like



In this case there is a unique ground state at  $\phi = 0$ . The Lagrangian is invariant under a  $\mathbb{Z}_2$  symmetry that takes  $\phi \rightarrow -\phi$ . This symmetry has the usual consequence: in the quantum theory an  $n$  particle state has parity  $(-1)^n$  under the symmetry, so the number of  $\phi$  quanta is conserved mod 2.

Now let's consider the same theory but with  $\mu^2 < 0$ . Then the potential looks like



Now there are two degenerate ground states located at  $\phi = \pm\sqrt{-\mu^2/\lambda}$ . Note that the  $\mathbb{Z}_2$  symmetry exchanges the two ground states. Taking  $\mu^2$  to be negative might bother you – does it mean the mass is imaginary? In a way it does: standard perturbation theory is an expansion about the unstable point  $\phi = 0$ , and the imaginary mass reflects the instability.

To see the physical consequences of having  $\mu^2 < 0$  it's best to expand the action about one of the degenerate minima. Just to be definite let's expand about the minimum on the right, and set

$$\phi = \phi_0 + \rho \quad \phi_0 = \sqrt{-\mu^2/\lambda}$$

Here  $\rho$  is a new field with the property that it vanishes in the appropriate ground state. Rewriting the action in terms of  $\rho$

$$\mathcal{L} = \frac{1}{2}\partial_\mu\rho\partial^\mu\rho - \frac{1}{2}(\phi_0 + \rho)^2 - \frac{1}{4}\lambda(\phi_0 + \rho)^4$$

Expanding this in powers of  $\rho$  there's a constant term (the value of  $V$  at its minimum) that we can ignore. The term linear in  $\rho$  vanishes since we're expanding about a minimum. We're left with

$$\mathcal{L} = \frac{1}{2}\partial_\mu\rho\partial^\mu\rho + \mu^2\rho^2 - \sqrt{-\mu^2}\lambda\rho^3 - \frac{1}{4}\lambda\rho^4 \quad (5.3)$$

The curious thing is that, if I just handed you this Lagrangian without telling you where it came from, you'd say that this is a theory of a real scalar field with

- a positive (mass)<sup>2</sup> given by  $m_\rho^2 = -2\mu^2 > 0$

- cubic and quartic self-couplings described by an interaction Hamiltonian  $\mathcal{H}_{\text{int}} = (-\mu^2\lambda)^{1/2}\rho^3 + \frac{1}{4}\lambda\rho^4$
- no sign of a  $\mathbb{Z}_2$  symmetry!

A few comments are in order.

- (i) A low-energy observer can only see small fluctuations about one of the degenerate minima. To such an observer the underlying  $\mathbb{Z}_2$  symmetry is not manifest, since it relates small fluctuations about one minimum to small fluctuations about the other minimum.
- (ii) The underlying symmetry is still valid, even if  $\mu^2 < 0$ , and it does have consequences at low energies. In particular the coefficient of the cubic term in the potential energy for  $\rho$  is not an independent coupling constant – it's fixed in terms of  $\lambda$  and  $m_\rho^2$ . So a low-energy observer who made very precise measurements of  $\rho - \rho$  scattering could deduce the existence of the other vacuum.
- (iii) A more straightforward way to discover the other vacuum is to work at high enough energies that the field can go over the barrier from one vacuum to the other.
- (iv) Although it's classically forbidden, in the quantum theory can't the field tunnel through the barrier to reach the other vacuum, even at low energies? The answer is no because, as you'll show on the homework, the tunneling probability vanishes for a quantum field in infinite spatial volume.

This last point is rather significant. In ordinary quantum mechanics states that are related by a symmetry can mix, and it's frequently the case that the ground state is unique and invariant under all symmetries.<sup>†</sup> But tunneling between different vacua is forbidden in quantum field theory in the infinite volume limit. This is what makes spontaneous symmetry breaking possible.

### 5.2.2 Breaking a continuous symmetry

Now let's consider a theory with a continuous symmetry. As a simple example, let's take a two-component real scalar field  $\vec{\phi}$  with

$$\mathcal{L} = \frac{1}{2}\partial_\mu\vec{\phi} \cdot \partial^\mu\vec{\phi} - \frac{1}{2}\mu^2|\vec{\phi}|^2 - \frac{1}{4}\lambda|\vec{\phi}|^4.$$

<sup>†</sup> For example the ground state of a particle in a double-well potential is unique, a symmetric combination of states localized in either well (Sakurai, *Modern quantum mechanics*, p. 256). Likewise the ground state of a rigid rotator has no angular momentum. This is not to say that in quantum mechanics the ground state is always unique. For example the ground state of a deuterium nucleus has total angular momentum  $J = 1$ .

This theory has an  $SO(2)$  symmetry

$$\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \mapsto \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$$

Noether's theorem gives the corresponding conserved current

$$j_\mu = \phi_2 \partial_\mu \phi_1 - \phi_1 \partial_\mu \phi_2.$$

First let's consider  $\mu^2 > 0$ . In this case there are no surprises. The potential has a unique minimum at  $\vec{\phi} = 0$ . The symmetry leaves the ground state invariant. The symmetry is manifest in the spectrum of small fluctuations (the particle spectrum): in particular

$$\phi_1 \text{ and } \phi_2 \text{ have the same mass.} \quad (5.4)$$

The conserved charge is also easy to interpret. For spatially homogeneous fields it's essentially the angular momentum you'd assign a particle rolling in the potential  $V(\vec{\phi})$ . Alternatively, if you work in terms of a complex field  $\Phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2)$ , then the conserved charge is the net number of  $\Phi$  quanta.

Now let's consider  $\mu^2 < 0$ . In this case there's a circle of degenerate minima located at  $|\vec{\phi}| = \sqrt{-\mu^2/\lambda}$ . Under the  $SO(2)$  symmetry these vacua get rotated into each other. To see what's going on let's introduce fields that are adapted to the symmetry, and set

$$\vec{\phi} = (\sigma \cos \theta, \sigma \sin \theta) \quad \sigma > 0, \quad \theta \approx \theta + 2\pi$$

In terms of  $\sigma$  and  $\theta$  the symmetry acts by

$$\sigma \text{ invariant,} \quad \theta \rightarrow \theta + \text{const.}$$

and we have

$$\mathcal{L} = \frac{1}{2} \partial_\mu \sigma \partial^\mu \sigma + \frac{1}{2} \sigma^2 \partial_\mu \theta \partial^\mu \theta - \frac{1}{2} \mu^2 \sigma^2 - \frac{1}{4} \lambda \sigma^4$$

When  $\mu^2 < 0$  the minimum of the potential is at  $\sigma = \sqrt{-\mu^2/\lambda}$ . To take this into account we shift

$$\sigma = \sqrt{-\mu^2/\lambda} + \rho$$

and find that (up to an additive constant)

$$\begin{aligned} \mathcal{L} = & \frac{1}{2} \partial_\mu \rho \partial^\mu \rho + \mu^2 \rho^2 - (-\mu^2 \lambda)^{1/2} \rho^3 - \frac{1}{4} \lambda \rho^4 \\ & + \frac{1}{2} (-\mu^2/\lambda) \partial_\mu \theta \partial^\mu \theta + (-\mu^2/\lambda)^{1/2} \rho \partial_\mu \theta \partial^\mu \theta + \frac{1}{2} \rho^2 \partial_\mu \theta \partial^\mu \theta \end{aligned}$$



The first line is, aside from the restriction  $\rho > 0$ , just the theory we encountered previously in (5.3): it describes a real scalar field with cubic and quartic self-couplings. The second line describes a scalar field  $\theta$  that has some peculiar-looking couplings to  $\rho$  but no mass term. (If you want you can redefine  $\theta$  to give it a canonical kinetic term.) If I didn't tell you where this Lagrangian came from you'd say that this is a theory with

- two scalar fields, one massive and the other massless
- cubic and quartic interactions between the fields
- no sign of any  $SO(2)$  symmetry!

Although there are some parallels with discrete symmetry breaking, there are also important differences. The main difference is that in the continuous case a massless scalar field  $\theta$  appears in the spectrum. It's easy to understand why it has to be there. The underlying  $SO(2)$  symmetry acts by shifting  $\theta \rightarrow \theta + \text{const.}$  The Lagrangian must be invariant under such a shift, which rules out any possible mass term for  $\theta$ .

One can make a stronger statement. The shift symmetry tells you that the potential energy is independent of  $\theta$ . So the symmetry forbids, not just a mass term, but any kind of non-derivative interaction for  $\theta$ . The usual terminology is that, once all other fields are set equal to their vacuum values,  $\theta$  parametrizes a flat direction in field space.

It's worth saying this again. We have a family of degenerate ground states labeled by the value of  $\theta$ . A low-energy observer could, in a localized region, hope to study a small fluctuation about one of these vacua. Unlike in the discrete case, a small fluctuation about one vacuum can reach some of the other "nearby" vacua. This is illustrated in Fig. 5.1. The energy density of such a fluctuation can be made arbitrarily small – even if the amplitude of the fluctuation is held fixed – just by making the wavelength of the fluctuation larger. This property, that the energy density goes to zero as the wavelength goes to infinity, manifests itself through the presence of a massless scalar field. These massless fields are known as Goldstone bosons.

Incidentally, suppose we were at such low energies that we couldn't create any  $\rho$  particles. Then we'd describe the dynamics just in terms of a (rescaled) Goldstone field  $\tilde{\theta} = \sqrt{-\mu^2/\lambda} \theta$  that takes values on a circle.

$$\mathcal{L} = \frac{1}{2} \partial_\mu \tilde{\theta} \partial^\mu \tilde{\theta} \quad \tilde{\theta} \approx \tilde{\theta} + 2\pi(-\mu^2/\lambda)^{1/2}$$

The circle should be thought of as the space of vacua of the theory. This is known as a low-energy effective action for the Goldstone boson.

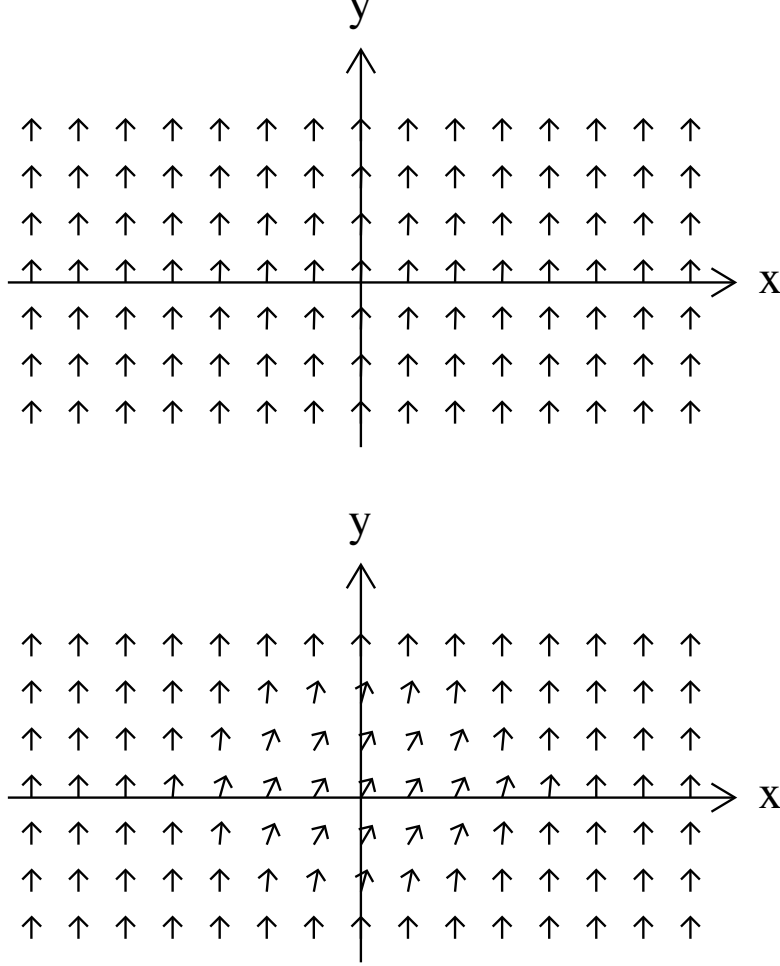


Fig. 5.1. A fluctuation in the model of section 5.2.2. The field  $\vec{\phi}(x, y, z = 0)$  is drawn as an arrow in the  $xy$  plane. Top figure: one of the degenerate vacuum states. Bottom figure: a low-energy fluctuation, in which the field in a certain region is slightly rotated. The energy density of such a fluctuation goes to zero as the wavelength increases.

### 5.2.3 Partially breaking a continuous symmetry

As a final example, let's consider the dynamics of a three-component real scalar field  $\vec{\phi}$  with the by now familiar-looking Lagrangian

$$\mathcal{L} = \frac{1}{2} \partial_\mu \vec{\phi} \cdot \partial^\mu \vec{\phi} - \frac{1}{2} \mu^2 |\vec{\phi}|^2 - \frac{1}{4} \lambda |\vec{\phi}|^4.$$

This theory has an  $SO(3)$  symmetry

$$\vec{\phi} \rightarrow R\vec{\phi} \quad R \in SO(3).$$

For  $\mu^2 > 0$  there's a unique vacuum at  $\vec{\phi} = 0$  and the  $SO(3)$  symmetry is unbroken. For  $\mu^2 < 0$  spontaneous symmetry breaking occurs.

When the symmetry is broken we have a collection of ground states characterized by

$$|\vec{\phi}| = \sqrt{-\mu^2/\lambda}.$$

That is, the space of vacua is a two-dimensional sphere  $S^2 \subset \mathbb{R}^3$ . The symmetry group acts on the sphere by rotations.

To proceed, let's first choose a vacuum to expand around. Without loss of generality we'll expand around the vacuum at the north pole of the sphere, namely the point

$$\vec{\phi} = (0, 0, \sqrt{-\mu^2/\lambda}).$$

Now let's see how our vacuum state behaves under symmetry transformations. Rotations in the 13 and 23 planes act non-trivially on our ground state. They move it to a different point on the sphere, thereby generating a two-dimensional space of flat directions. Corresponding to this we expect to find two massless Goldstone bosons in the spectrum. Rotations in the 12 plane, however, leave our choice of vacuum invariant. They form an unbroken  $SO(2)$  subgroup of the underlying  $SO(3)$  symmetry.

To make this a bit more concrete it's convenient to parametrize fields near the north pole of the sphere in terms of three real scalar fields  $\sigma, x, y$  defined by

$$\vec{\phi} = \sigma \left( x, y, \sqrt{1 - x^2 - y^2} \right). \quad (5.5)$$

The field  $\sigma$  parametrizes radial fluctuations in the fields, while  $x$  and  $y$  parametrize points on a unit two-sphere. As in our previous examples one can set  $\sigma = \sqrt{-\mu^2/\lambda} + \rho$  and find that  $\rho$  has a mass  $m_\rho^2 = -2\mu^2$ . The fields  $x$  and  $y$  are Goldstone bosons. Substituting (5.5) into the Lagrangian and setting  $\rho = 0$  one is left with the low-energy effective action for the Goldstone bosons

$$\mathcal{L} = \frac{1}{2} \left( \frac{-\mu^2}{\lambda} \right) \frac{1}{1 - x^2 - y^2} \left( \partial_\mu x \partial^\mu x + \partial_\mu y \partial^\mu y - (x \partial_\mu y - y \partial_\mu x)(x \partial^\mu y - y \partial^\mu x) \right).$$

Note that the Goldstone bosons  $\begin{pmatrix} x \\ y \end{pmatrix}$  transform as a doublet of the unbroken  $SO(2)$  symmetry.

### 5.2.4 Symmetry breaking in general

Many aspects of symmetry breaking are determined purely by group theory. Consider a theory with a symmetry group  $G$ . Suppose we've found a ground state where the fields (there could be more than one) take on a value I'll denote  $\phi_0$ . The theory might not have a unique ground state. If we act on  $\phi_0$  with some  $g \in G$  we must get another state with exactly the same energy. This means  $g\phi_0$  is also a ground state. Barring miracles we'd expect to obtain the entire space of vacua in this way:

$$\mathcal{M} = (\text{space of vacua}) = \{g\phi_0 : g \in G\}$$

Now it's possible that some (or all) elements of  $G$  leave the vacuum  $\phi_0$  invariant. That is, there could be a subgroup  $H \subset G$  such that

$$h\phi_0 = \phi_0 \quad \forall h \in H.$$

In this case  $H$  survives as an unbroken symmetry group. One says that  $G$  is spontaneously broken to  $H$ .

This leads to a nice representation of the space of vacua. If  $g_1 = g_2h$  for some  $h \in H$  then  $g_1$  and  $g_2$  have exactly the same effect on  $\phi_0$ :  $g_1\phi_0 = g_2\phi_0$ . This means that  $\mathcal{M}$  is actually a quotient space,  $\mathcal{M} = G/H$ , where the notation just means we've imposed an equivalence relation:

$$G/H \equiv G/\{g_1 \sim g_2 \text{ if } g_1 = g_2h \text{ for some } h \in H\}.$$

This also leads to a nice geometrical picture of the Goldstone bosons: they are simply fields  $\phi^I$  which parametrize  $\mathcal{M}$ . One can be quite explicit about the general form of the action for the Goldstone fields. If you think of the space of vacua as a manifold  $\mathcal{M}$  with coordinates  $\phi^I$  and metric  $G_{IJ}(\phi)$ , the low-energy effective action is

$$S = \int d^4x \frac{1}{2} G_{IJ}(\phi) \partial_\mu \phi^I \partial^\mu \phi^J.$$

Actions of this form are known as “non-linear  $\sigma$ -models.”<sup>†</sup> At the classical level, to find the metric  $G_{IJ}$  one can proceed as we did in our  $SO(2)$  example: rewrite the underlying Lagrangian in terms of Goldstone fields which parametrize the space of vacua (the analogs of  $\theta$ ) together with fields that parametrize “radial directions” (the analogs of  $\rho$ ). Setting the radial fields equal to their vacuum values, one is left with a non-linear  $\sigma$ -model for the

<sup>†</sup> Obscure terminology, referring to the fact that the Goldstone fields take values in a curved space. If the space of vacua is embedded in a larger linear space, say  $\mathcal{M} \subset \mathbb{R}^n$ , then the action for the fields that parametrize the embedding space  $\mathbb{R}^n$  is known as a linear  $\sigma$ -model.

Goldstone fields, and one can read off the metric from the action.<sup>‡</sup> Finally, the number of Goldstone bosons is equal to the number of broken symmetry generators, or equivalently the dimension of the quotient space:

$$\# \text{ Goldstones} = \dim \mathcal{M} = \dim G - \dim H.$$

### 5.3 Spontaneous symmetry breaking (quantum)

Now let's see more directly how spontaneous symmetry breaking plays out in the quantum theory. First we need to decide: what will signal spontaneous symmetry breaking? To this end let's assume we have a collection of degenerate ground states related by a symmetry. Let me denote two of these ground states  $|0\rangle$ ,  $|0'\rangle$ . The symmetry is *spontaneously broken* if  $|0\rangle \neq |0'\rangle$ .

Instead of applying this criterion directly, it's often more convenient to search for a field  $\phi$  whose vacuum expectation value transforms under the symmetry:

$$\langle 0|\phi|0\rangle \neq \langle 0'|\phi|0'\rangle.$$

This implies  $|0\rangle \neq |0'\rangle$ , so it implies spontaneous symmetry breaking as defined above. Such expectation values are known as “order parameters.”

Now let's specialize to continuous symmetries. In this case we have a conserved current  $j^\mu(t, \mathbf{x})$ , and we can construct a unitary operator

$$U(\lambda) = e^{-i \int d^3x \lambda(\mathbf{x}) j^0(t, \mathbf{x})}$$

which implements a position-dependent symmetry transformation parametrized by  $\lambda(\mathbf{x})$ . For infinitesimal  $\lambda$ , the change in the ground state  $|0\rangle$  is

$$\delta|0\rangle = -i \int d^3x \lambda(\mathbf{x}) j^0(t, \mathbf{x}) |0\rangle$$

The condition for spontaneous symmetry breaking is that  $\delta|0\rangle$  does not vanish, even as  $\lambda$  approaches a constant.<sup>†</sup>

Claim: for each broken symmetry there is a massless Goldstone boson in the spectrum.

Let's give an intuitive proof of this fact. The symmetry takes us from

<sup>‡</sup> For instance, in our  $SO(3)$  example,  $\frac{1}{1-x^2-y^2} \begin{pmatrix} 1-y^2 & xy \\ xy & 1-x^2 \end{pmatrix}$  is the metric on a unit two-sphere.

<sup>†</sup> Formally as  $\lambda(\mathbf{x})$  approaches a constant we have  $\delta|0\rangle = -i\lambda Q|0\rangle$ , where  $Q = \int d^3x j^0$  is the generator of the symmetry. So the condition for spontaneous symmetry breaking is  $Q|0\rangle \neq 0$ . However one should be careful about discussing  $Q$  for a spontaneously broken symmetry: see Burgess and Moore, exercise 8.2 or Ryder, *Quantum field theory* p. 300.

one choice of vacuum state to another, at no cost in energy. Therefore a small, long-wavelength fluctuation in our choice of vacuum will cost very little energy. We can write down a state corresponding to a fluctuating choice of vacuum quite explicitly: it's just the state

$$U(\lambda)|0\rangle$$

produced by acting on  $|0\rangle$  with the unitary operator  $U(\lambda)$ . Setting  $\lambda = e^{i\mathbf{p}\cdot\mathbf{x}}$ , where  $\mathbf{p}$  is a 3-vector that determines the spatial wavelength of the fluctuation, to first order the change in the ground state is

$$\delta|0\rangle \equiv |\mathbf{p}\rangle = -i \int d^3x e^{i\mathbf{p}\cdot\mathbf{x}} j^0(t, \mathbf{x}) |0\rangle.$$

The state we've defined satisfies two properties:

1. It represents an excitation with spatial 3-momentum  $\mathbf{p}$ .
2. The energy of the excitation vanishes as  $\mathbf{p} \rightarrow 0$ .

This shows that  $|\mathbf{p}\rangle$  describes a massless particle. We identify it as the state representing a single Goldstone boson with 3-momentum  $\mathbf{p}$ . To establish the above properties, note that

1. Under a spatial translation  $\mathbf{x} \rightarrow \mathbf{x} + \mathbf{a}$  the state  $|\mathbf{p}\rangle$  transforms the way a momentum eigenstate should: acting with a spatial translation operator  $T_{\mathbf{a}}$  we get

$$T_{\mathbf{a}}|\mathbf{p}\rangle = \int d^3x e^{i\mathbf{p}\cdot\mathbf{x}} j^0(t, \mathbf{x} + \mathbf{a}) |0\rangle = e^{-i\mathbf{p}\cdot\mathbf{a}} |\mathbf{p}\rangle.$$

2. We argued above that  $|0\rangle$  and  $U(\lambda)|0\rangle \approx |0\rangle + \delta|0\rangle$  become degenerate in energy as the wavelength of the fluctuation increases. This means that for large wavelength  $\delta|0\rangle$  has the same energy as  $|0\rangle$  itself. So the energy associated with the excitation  $|\mathbf{p}\rangle$  must vanish as  $\mathbf{p} \rightarrow 0$ .<sup>†</sup>

This argument shows that the broken symmetry currents create Goldstone bosons from the vacuum.<sup>‡</sup> This can be expressed in the Lorentz-invariant form

$$\langle \pi(p) | j^\mu(x) | 0 \rangle = i f p^\mu e^{ip \cdot x}$$

where  $|\pi(p)\rangle$  is an on-shell one-Goldstone-boson state with 4-momentum  $p$  and  $f$  is a fudge factor to normalize the state.<sup>§</sup> Note that current conser-

<sup>†</sup> What would happen if we tried to carry out this construction with an unbroken symmetry generator, i.e. one that satisfies  $Q|0\rangle = 0$ ?

<sup>‡</sup> For a rigorous proof of this see Weinberg, *Quantum theory of fields*, vol. II p. 169 – 173.

<sup>§</sup> For reasons you'll see in problem 8.2, for pions the fudge factor  $f$  is known as the pion decay constant. It has the numerical value  $f_\pi = 93 \text{ MeV}$ .

vation  $\partial_\mu j^\mu = 0$  implies that  $p^2 = 0$ , i.e. massless Goldstone bosons. More generally, if we had multiple broken symmetry generators  $Q^a$  we'd have

$$\langle \pi^a(p) | j^{\mu b}(x) | 0 \rangle = i f^{ab} p^\mu e^{ip \cdot x} \quad (5.6)$$

i.e. one Goldstone boson for each broken symmetry generator.

Finally, we can see the loophole in the usual argument that symmetries  $\Rightarrow$  degeneracies in the spectrum. Let  $Q^a$  be a symmetry generator and let  $\phi_i$  be a collection of fields that form a representation of the symmetry:  $i[Q^a, \phi_i] = \mathcal{D}^a_i{}^j \phi_j$ . Then

$$iQ^a \phi_i | 0 \rangle = i[Q^a, \phi_i] | 0 \rangle + i\phi_i Q^a | 0 \rangle = \mathcal{D}^a_i{}^j \phi_j | 0 \rangle + i\phi_i Q^a | 0 \rangle.$$

The first term is standard: by itself it says particle states form a representation of the symmetry group. But when the symmetry is spontaneously broken the second term is non-zero.

## References

Symmetries and spontaneous symmetry breaking are discussed in Cheng & Li sections 5.1 and 5.3. There's some nice discussion in Quigg sections 2.3, 5.1, 5.2. Peskin & Schroeder discuss symmetries in section 2.2 and spontaneous symmetry breaking in section 11.1.

## Exercises

### 5.1 Noether's theorem for fermions

Consider a general Lagrangian  $\mathcal{L}(\psi, \partial_\mu \psi)$  for a fermionic field  $\psi$ . To incorporate Fermi statistics  $\psi$  should be treated as an anticommuting or Grassmann-valued number. Recall that Grassmann numbers behave like ordinary numbers except that multiplication anti-commutes: if  $a$  and  $b$  are two Grassmann numbers then  $ab = -ba$ . One can define differentiation in the obvious way; if  $a$  and  $b$  are independent Grassmann variables then

$$\frac{\partial a}{\partial a} = \frac{\partial b}{\partial b} = 1 \quad \frac{\partial a}{\partial b} = \frac{\partial b}{\partial a} = 0.$$

The derivative operators themselves are anticommuting quantities. When differentiating products of Grassmann variables we need to

be careful about ordering. For example we can define a derivative operator that acts from the left, satisfying

$$\frac{\partial^L}{\partial a^L}(ab) = \frac{\partial a}{\partial a}b - a\frac{\partial b}{\partial a} = b,$$

or one that acts from the right, satisfying

$$\frac{\partial^R}{\partial a^R}(ab) = a\frac{\partial b}{\partial a} - \frac{\partial a}{\partial a}b = -b.$$

- (i) Show that the Euler-Lagrange equations which make the action for  $\psi$  stationary are

$$\partial_\mu \frac{\partial^R \mathcal{L}}{\partial (\partial_\mu \psi)^R} - \frac{\partial^R \mathcal{L}}{\partial \psi^R} = 0.$$

- (ii) Suppose the Lagrangian is invariant under an infinitesimal transformation  $\psi \rightarrow \psi + \delta\psi$ . Show that the current

$$j^\mu = \frac{\partial^R \mathcal{L}}{\partial (\partial_\mu \psi)^R} \delta\psi$$

is conserved. You should treat  $\delta\psi$  as a Grassmann number.

## 5.2 Symmetry breaking in finite volume?

Consider the quantum mechanics of a particle moving in a double well potential, described by the Lagrangian

$$L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}m\omega^2 x^2 - \frac{1}{4}m\lambda x^4.$$

We're taking the parameter  $\omega^2$  to be negative.

- (i) Expand the Lagrangian to quadratic order about the two minima of the potential, and write down approximate (harmonic oscillator) ground state wavefunctions

$$\begin{aligned}\Psi_+(x) &= \langle x|+\rangle \\ \Psi_-(x) &= \langle x|-\rangle\end{aligned}$$

describing unit-normalized states  $|+\rangle$  and  $|-\rangle$  localized in the right and left wells, respectively. How do your wavefunctions behave as  $m \rightarrow \infty$ ?

- (ii) Use the WKB approximation to estimate the tunneling amplitude  $\langle -|+\rangle$ . You can make approximations which are valid for large  $m$  (equivalently small  $\lambda$ ).



Now consider a real scalar field with Lagrangian density ( $\mu^2 < 0$ )

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} \mu^2 \phi^2 - \frac{1}{4} \lambda \phi^4.$$

The  $\phi \rightarrow -\phi$  symmetry is supposed to be spontaneously broken, with two degenerate ground states  $|+\rangle$  and  $|-\rangle$ . But can't the field tunnel from one minimum to the other? To see what's happening consider the same theory but in a finite spatial volume. For simplicity let's work in a spatial box of volume  $V$  with periodic boundary conditions, so that we can expand the field in spatial Fourier modes

$$\phi(t, \mathbf{x}) = \sum_{\mathbf{k}} \phi_{\mathbf{k}}(t) e^{i\mathbf{k} \cdot \mathbf{x}} \quad \phi_{-\mathbf{k}}(t) = \phi_{\mathbf{k}}^*(t).$$

Here  $\mathbf{k} = 2\pi \mathbf{n}/V^{1/3}$  with  $\mathbf{n} \in \mathbb{Z}^3$ .

- (iii) Expand the field theory Lagrangian  $L = \int d^3x \mathcal{L}$  to quadratic order about the classical vacua. Express your answer in terms of the Fourier coefficients  $\phi_{\mathbf{k}}(t)$  and their time derivatives.
- (iv) Use the Lagrangian worked out in part (iii) to write down approximate ground state wavefunctions

$$\begin{array}{ll} \Psi_+(\phi_{\mathbf{k}}) & \text{describing } |+\rangle \\ \Psi_-(\phi_{\mathbf{k}}) & \text{describing } |-\rangle \end{array}$$

How do your wavefunctions behave in the limit  $V \rightarrow \infty$ ?

- (v) If you neglect the coupling between different Fourier modes – something which should be valid at small  $\lambda$  – then the Lagrangian for the constant mode  $\phi_{\mathbf{k}=0}$  should look familiar. Use your quantum mechanics results to estimate the tunneling amplitude  $\langle -|+\rangle$  between the two (unit normalized) ground states. How does your result behave as  $V \rightarrow \infty$ ?
- (vi) How do matrix elements of any finite number of field operators between the left and right vacua  $\langle -|\phi(t_1, \mathbf{x}_1) \cdots \phi(t_n, \mathbf{x}_n)|+\rangle$  behave as  $V \rightarrow \infty$ ?

Moral of the story: spontaneous symmetry breaking is a phenomenon associated with the thermodynamic ( $V \rightarrow \infty$ ) limit. For a nice discussion of this see Weinberg QFT vol. II sect. 19.1.

5.3  $O(N)$  linear  $\sigma$ -model

Consider the Lagrangian

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \cdot \partial^\mu \phi - \frac{1}{2} \mu^2 |\phi|^2 - \frac{1}{4} \lambda |\phi|^4$$

Here  $\phi$  is a vector containing  $N$  scalar fields. Note that  $\mathcal{L}$  is invariant under rotations  $\phi \rightarrow R\phi$  where  $R \in SO(N)$ .

- (i) Find the conserved currents associated with this symmetry.
- (ii) When  $\mu^2 < 0$  the  $SO(N)$  symmetry is spontaneously broken. In this case identify
  - the space of vacua
  - the unbroken symmetry group
  - the spectrum of particle masses

5.4  $O(4)$  linear  $\sigma$ -model

Specialize to  $N = 4$  and define  $\Sigma = \phi_4 \mathbf{1} + \sum_{k=1}^3 i\phi_k \tau_k$  where  $\tau_k$  are Pauli matrices.

- (i) Show that  $\det \Sigma = |\phi|^2$  and  $\Sigma^* = \tau_2 \Sigma \tau_2$ .
- (ii) Rewrite  $\mathcal{L}$  in terms of  $\Sigma$ .
- (iii) In place of  $SO(4)$  transformations on  $\phi$  we now have  $SU(2)_L \times SU(2)_R$  transformations on  $\Sigma$ . These transformations act by  $\Sigma \rightarrow L\Sigma R^\dagger$  where  $L, R \in SU(2)$ . Show that these transformations leave  $\det \Sigma$  invariant and preserve the property  $\Sigma^* = \tau_2 \Sigma \tau_2$ .
- (iv) Show that one can set  $\Sigma = \sigma U$  where  $\sigma > 0$  and  $U \in SU(2)$ .
- (v) Rewrite the Lagrangian in terms of  $\sigma$  and  $U$ . Take  $\mu^2 < 0$  so the  $SU(2)_L \times SU(2)_R$  symmetry is spontaneously broken and, in terms of the fields  $\sigma$  and  $U$ , identify
  - the space of vacua
  - the unbroken symmetry group
  - the spectrum of particle masses
- (vi) Write down the low energy effective action for the Goldstone bosons.

5.5  $SU(N)$  **nonlinear  $\sigma$ -model**

Consider the Lagrangian  $\mathcal{L} = \frac{1}{4}f^2 \text{Tr} (\partial_\mu U^\dagger \partial^\mu U)$  where  $f$  is a constant with units of  $(\text{mass})^2$  and  $U \in SU(N)$ . The Lagrangian is invariant under  $U \rightarrow LUR^\dagger$  where  $L, R \in SU(N)$ . Identify

- the space of vacua
- the unbroken symmetry group
- the spectrum of particle masses

## 6

### Chiral symmetry breaking

Physics 85200

January 6, 2014

Now we're ready to see how some of these ideas of symmetries and symmetry breaking are realized by the strong interactions. But first, some terminology. If one can decompose

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1$$

where  $\mathcal{L}_0$  is invariant under a symmetry and  $\mathcal{L}_1$  is non-invariant but can be treated as a perturbation, then one has “explicit symmetry breaking” by a term in the Lagrangian. This is to be contrasted with “spontaneous symmetry breaking,” where the Lagrangian is invariant but the ground state is not. Incidentally, one can have both spontaneous and explicit symmetry breaking, if  $\mathcal{L}_0$  by itself breaks the symmetry spontaneously while  $\mathcal{L}_1$  breaks it explicitly.

Let's return to the quark model of section 3.1. For the time being we'll ignore quark masses. With three flavors of quarks assembled into

$$\psi = \begin{pmatrix} u \\ d \\ s \end{pmatrix}$$

we guessed that the strong interaction Lagrangian looked like

$$\mathcal{L}_{\text{strong}} = \bar{\psi} i \gamma^\mu \partial_\mu \psi + \dots$$

As discussed in section 5.1.1 the quark kinetic terms have an  $SU(3)$  symmetry  $\psi \rightarrow U\psi$ . Assuming this symmetry extends to all of  $\mathcal{L}_{\text{strong}}$  the corresponding conserved currents are

$$j^{\mu a} = \bar{\psi} \gamma^\mu T^a \psi$$

where the generators  $T^a$  are  $3 \times 3$  traceless Hermitian matrices.

In fact the quark kinetic terms have a larger symmetry group. To make this manifest we need to decompose the Dirac spinors  $u, d, s$  into their left- and right-handed chiral components. The calculation is identical to what we did for QED in section 4.1. The result is

$$\mathcal{L}_{\text{strong}} = \bar{\psi}_L i\gamma^\mu \partial_\mu \psi_L + \bar{\psi}_R i\gamma^\mu \partial_\mu \psi_R + \dots$$

Here

$$\psi_L = \frac{1}{2}(1 - \gamma^5)\psi \quad \text{and} \quad \psi_R = \frac{1}{2}(1 + \gamma^5)\psi$$

are 4-component spinors, although only two of their components are non-zero, and

$$\bar{\psi}_L \equiv (\psi_L)^\dagger \gamma^0 \quad \bar{\psi}_R \equiv (\psi_R)^\dagger \gamma^0.$$

This chiral decomposition makes it clear that the quark kinetic terms actually have an  $SU(3)_L \times SU(3)_R$  symmetry that acts independently on the left- and right-handed chiral components.<sup>†</sup>

$$\psi_L \rightarrow L\psi_L \quad \psi_R \rightarrow R\psi_R \quad L, R \in SU(3) \quad (6.1)$$

It's easy to work out the corresponding conserved currents; they're just what we had above except they only involve one of the chiral components:

$$j_L^{\mu a} = \bar{\psi}_L \gamma^\mu T^a \psi_L = \bar{\psi} \gamma^\mu T^a \frac{1}{2}(1 - \gamma^5)\psi$$

$$j_R^{\mu a} = \bar{\psi}_R \gamma^\mu T^a \psi_R = \bar{\psi} \gamma^\mu T^a \frac{1}{2}(1 + \gamma^5)\psi$$

It's often convenient to work in terms of the “vector” and “axial-vector” combinations

$$j_V^{\mu a} = j_L^{\mu a} + j_R^{\mu a} = \bar{\psi} \gamma^\mu T^a \psi$$

$$j_A^{\mu a} = -j_L^{\mu a} + j_R^{\mu a} = \bar{\psi} \gamma^\mu \gamma^5 T^a \psi$$

The question is what to make of this larger symmetry group. As we've seen the vector current corresponds to Gell-Mann's flavor  $SU(3)$ . But what about the axial current?

The simplest possibility would be for  $SU(3)_A$  to be explicitly broken by  $\mathcal{L}_{\text{strong}}$ : after all we've only been looking at the quark kinetic terms.<sup>‡</sup> I can't

<sup>†</sup> The full symmetry is  $U(3)_L \times U(3)_R$ . As we've seen the extra vector-like  $U(1)$  corresponds to conservation of baryon number. The fate of the extra axial  $U(1)$  is a fascinating story we'll return to in section 13.3.

<sup>‡</sup> Picky, picky: the symmetry group is really  $SU(3)_L \times SU(3)_R$ . The linear combination  $R - L$  that appears in the axial current doesn't generate a group, since two axial charges commute to give a vector charge. I'll call the axial symmetries  $SU(3)_A$  anyways.

say anything against this possibility, except that we might as well assume  $SU(3)_A$  is a valid symmetry and see where that assumption leads.

Another possibility is for  $SU(3)_A$  to be a manifest symmetry of the particle spectrum. We can rule this out right away. The axial charges

$$Q_A^a = \int d^3x j_A^{0a}$$

are odd under parity (see Peskin & Schroeder p. 65), so they change the parity of any state they act on. If  $SU(3)_A$  were a manifest symmetry there would have to be scalar (as opposed to pseudoscalar) particles with the same mass as the pions.

So we're left with the idea that  $SU(3)_A$  is a valid symmetry of the strong interaction Lagrangian, but is spontaneously broken by a choice of ground state. What order parameter could signal symmetry breaking? It's a bit subtle, but suppose the fermion bilinear  $\psi\bar{\psi}$  acquires an expectation value:

$$\langle 0|\psi\bar{\psi}|0\rangle = \mu^3 \mathbf{1}_{\text{flavor}} \otimes \mathbf{1}_{\text{spin}}.$$

Here  $\mu$  is a constant with dimensions of mass, and  $\mathbf{1}$  represents the identity matrix either in flavor or spinor space. In terms of the chiral components  $\psi_L, \psi_R$  this is equivalent to

$$\begin{aligned}\langle \psi_L \bar{\psi}_R \rangle &= \mu^3 \mathbf{1}_{\text{flavor}} \otimes \frac{1}{2}(1 - \gamma^5)_{\text{spin}} \\ \langle \psi_R \bar{\psi}_L \rangle &= \mu^3 \mathbf{1}_{\text{flavor}} \otimes \frac{1}{2}(1 + \gamma^5)_{\text{spin}} \\ \langle \psi_L \bar{\psi}_L \rangle &= \langle \psi_R \bar{\psi}_R \rangle = 0.\end{aligned}\tag{6.2}$$

What's nice is that this expectation value

- is invariant under Lorentz transformations (check!)
- is invariant under  $SU(3)_V$  transformations  $\psi_L \rightarrow U\psi_L, \psi_R \rightarrow U\psi_R$
- completely breaks the  $SU(3)_A$  symmetry

That is, in (6.1) one needs to set  $L = R$  in order to preserve the expectation value (6.2). So the claim is that strong-coupling effects in QCD cause  $q\bar{q}$  pairs to condense out of the trivial (perturbative) vacuum; the “chiral condensate” (6.2) is supposed to be generated dynamically by the strong interactions.<sup>†</sup>

In fact, the expectation value can be a bit more general. Whenever a continuous symmetry is spontaneously broken there should be a manifold

<sup>†</sup> People often characterize the strength of the chiral condensate by the spinor trace of (6.2), namely  $\langle \bar{u}u \rangle = \langle \bar{d}d \rangle = \langle \bar{s}s \rangle = -4\mu^3$  where the  $-$  sign arises from Fermi statistics.

of inequivalent vacua. We can find this space of vacua just by applying  $SU(3)_L \times SU(3)_R$  transformations to the vev (6.2). The result is

$$\langle \psi_L \bar{\psi}_R \rangle = \mu^3 U \otimes \frac{1}{2}(1 - \gamma^5) \quad \langle \psi_R \bar{\psi}_L \rangle = \mu^3 U^\dagger \otimes \frac{1}{2}(1 + \gamma^5) \quad \langle \psi_L \bar{\psi}_L \rangle = \langle \psi_R \bar{\psi}_R \rangle = 0$$

where  $U = LR^\dagger$  is an  $SU(3)$  matrix. In terms of Dirac spinors this can be rewritten as

$$\langle 0 | \psi \bar{\psi} | 0 \rangle = \mu^3 e^{-i\lambda^a T^a \gamma^5} \quad (6.3)$$

where  $U = e^{i\lambda^a T^a}$ . If our conjecture is right, the space of vacua of QCD is labeled by an  $SU(3)$  matrix  $U$ . We'd expect to have  $\dim SU(3) = 8$  massless Goldstone bosons that can be described by a field  $U(t, \mathbf{x})$ . If we're at very low energies then the dynamics of QCD reduces to an effective theory of the Goldstone bosons. What could the action be? As we'll discuss in more detail in the next chapter, there's a unique candidate with at most two derivatives: the non-linear  $\sigma$ -model action from the last homework!

$$\mathcal{L}_{\text{eff}} = \frac{1}{4} f^2 \text{Tr} \left( \partial_\mu U^\dagger \partial^\mu U \right)$$

This action provides a complete description of the low-energy dynamics of QCD with three massless quarks.

In the real world various effects – in particular quark mass terms – explicitly break chiral symmetry. To get an idea of the consequences, current estimates are that the chiral condensate is characterized by  $\mu \approx 160 \text{ MeV}$ . This is large compared to the light quark masses

$$m_u \approx 3 \text{ MeV} \quad m_d \approx 5 \text{ MeV} \quad m_s \approx 100 \text{ MeV}$$

but small compared to the heavy quark masses

$$m_c \approx 1.3 \text{ GeV} \quad m_b \approx 4.2 \text{ GeV} \quad m_t \approx 172 \text{ GeV}.$$

For the light quarks the explicit breaking can be treated as a small perturbation of the chiral condensate, so the strong interactions have an approximate  $SU(3)_L \times SU(3)_R$  symmetry. The explicit breaking turns out to give a small mass to the would-be Goldstone bosons that arise from spontaneous  $SU(3)_A$  breaking. Thus in the real world we expect to find eight anomalously light scalar particles which we can identify with  $\pi$ ,  $K$ ,  $\eta$ . This explains why the octet mesons are so light – they're approximate Goldstone bosons! This also explains Gell-Mann's flavor  $SU(3)$  symmetry and shows why there is no useful larger flavor symmetry.<sup>†</sup>

<sup>†</sup> In this discussion we assumed that  $\mu$  sets the relevant energy scale. To justify  $SU(3)_A$  as an approximate symmetry, it would really be more appropriate to compare the octet meson

### References

Chiral symmetry breaking is discussed in Cheng & Li sections 5.4 and 5.5, but using a rather old-fashioned algebraic approach. Peskin & Schroeder discuss chiral symmetry breaking on pages 667 – 670.

### Exercises

#### 6.1 Vacuum alignment in the $\sigma$ -model

Suppose we add an explicit symmetry-breaking perturbation to our  $O(4)$  linear  $\sigma$ -model Lagrangian of problem 5.4.

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \cdot \partial^\mu \phi - \frac{1}{2} \mu^2 |\phi|^2 - \frac{1}{4} \lambda |\phi|^4 + \mathbf{a} \cdot \phi$$

Here  $\mu^2 < 0$  and  $\mathbf{a}$  is a constant vector; for simplicity you can take it to point in the  $\phi_4$  direction. What is the unbroken symmetry group? Identify the (unique) vacuum state and expand about it by setting

$$\Sigma = (f + \rho) e^{i\pi \cdot \tau / f}$$

Here  $f$  is a constant and  $\rho$  and  $\pi$  are fields with  $\langle \rho \rangle = \langle \pi \rangle = 0$ . Identify the spectrum of particle masses.

#### 6.2 Vacuum alignment in QCD

Strong interactions are supposed to generate a non-zero expectation value that spontaneously breaks  $SU(3)_L \times SU(3)_R \rightarrow SU(3)_V$ . The space of vacua can be parametrized by a unitary matrix  $U = e^{i\lambda^a T^a}$  that characterizes the expectation value

$$\langle \psi \bar{\psi} \rangle = \mu^3 e^{-i\lambda^a T^a \gamma^5}.$$

Here  $\mu$  is a constant with dimensions of mass. The low energy effective Lagrangian for the resulting Goldstone bosons is

$$\mathcal{L} = \frac{1}{4} f^2 \text{Tr}(\partial_\mu U^\dagger \partial^\mu U)$$

where  $f$  is another constant with dimensions of mass.

- (i) Consider adding a quark mass term  $\mathcal{L}_{\text{mass}} = -\bar{\psi} M \psi$  to the underlying strong interaction Lagrangian. Argue that for small quark

masses, which measure the strength of  $SU(3)_A$  breaking, to the scale of chiral perturbation theory discussed on p. 80.



masses the explicit breaking due to the mass term can be taken into account by modifying the effective Lagrangian to read

$$\frac{1}{4}f^2\text{Tr}(\partial_\mu U^\dagger \partial^\mu U) + 2\mu^3\text{Tr}(M(U + U^\dagger)).$$

- (ii) Identify the ground state of the resulting theory. Compute the matrix of would-be Goldstone boson masses by expanding the action to quadratic order in the fields  $\pi^a$ , where  $\pi^a$  is defined by  $U = e^{i\pi^a T^a/f}$  with  $\text{Tr} T^a T^b = 2\delta^{ab}$ .
- (iii) Use your results to predict the  $\eta$  mass in terms of  $m_{\pi^\pm}^2$ ,  $m_{\pi^0}^2$ ,  $m_{K^\pm}^2$ ,  $m_{K^0}^2$ ,  $m_{\bar{K}^0}^2$ . How does your prediction compare to the data? (You can ignore small isospin breaking effects and set  $m_u = m_d$ .)

## 7

# Effective field theory and renormalization

Physics 85200

January 6, 2014

### 7.1 Effective field theory

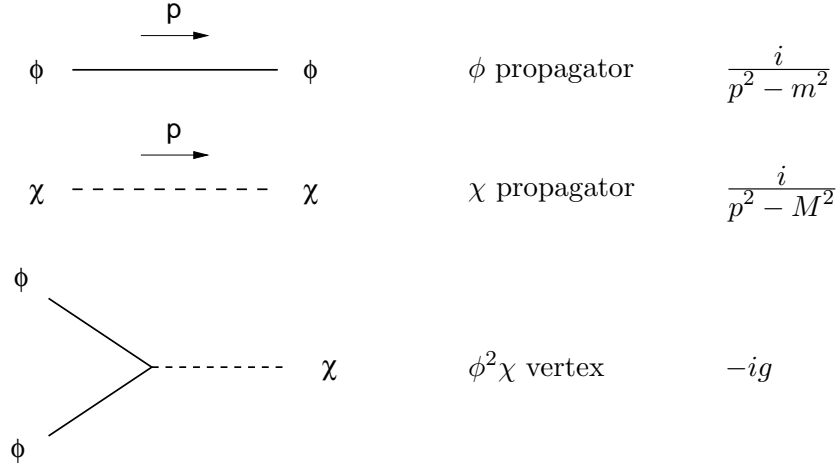
In a couple places – deriving  $SU(3)$  symmetry currents of the quark model, writing down effective actions for Goldstone bosons – we’ve given arguments involving dimensional analysis and the notion of an approximate low-energy description. I’d like to discuss these ideas a little more explicitly. I’ll proceed by way of two examples.

#### 7.1.1 Example I: $\phi^2\chi$ theory

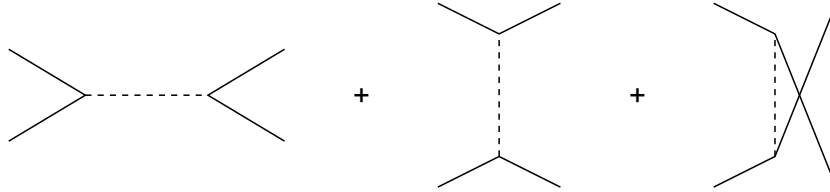
Let me start with the following Lagrangian for two scalar fields.

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 + \frac{1}{2}\partial_\mu\chi\partial^\mu\chi - \frac{1}{2}M^2\chi^2 - \frac{1}{2}g\phi^2\chi \quad (7.1)$$

Here  $g$  is a coupling with dimensions of mass. We’ll be interested in  $m \ll M$  with  $g$  and  $M$  comparable in magnitude. To be concrete let’s study  $\phi$ - $\phi$  scattering. The Feynman rules are



At lowest order the diagrams are



$$\Rightarrow \mathcal{M} = \frac{g^2}{s - M^2} + \frac{g^2}{t - M^2} + \frac{g^2}{u - M^2}$$

Here  $s = (p_1 + p_2)^2$ ,  $t = (p_1 - p_3)^2$ ,  $u = (p_1 - p_4)^2$  are the usual Mandelstam variables.

Suppose a low-energy observer sets out to study  $\phi$ - $\phi$  scattering at a center-of-mass energy  $E = \sqrt{s} \ll M$ . Such an observer can't directly detect  $\chi$  particles. To understand what such an observer does see, let's expand the scattering amplitude in inverse powers of  $M$  (recall that we're counting  $g = \mathcal{O}(M)$ ):

$$\mathcal{M} = \underbrace{-\frac{3g^2}{M^2}}_{\mathcal{O}(M^0)} - \underbrace{\frac{g^2(s+t+u)}{M^4}}_{\mathcal{O}(1/M^2)} - \underbrace{\frac{g^2(s^2+t^2+u^2)}{M^6}}_{\mathcal{O}(1/M^4)} + \dots \quad (7.2)$$

How would our low-energy observer interpret this expansion?

At leading order the scattering amplitude is simply  $-3g^2/M^2$ . A low-energy observer would interpret this as coming from an elementary  $\phi^4$  in-

teraction – that is, in terms of an “effective Lagrangian”

$$\mathcal{L}_4 = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 - \frac{1}{4!} \lambda \phi^4.$$

$\mathcal{L}_4$  is known as a dimension-4 effective Lagrangian since it includes operators with (mass) dimension up to 4. This reproduces the leading term in the  $\phi$ - $\phi$  scattering amplitude provided  $\lambda = -3g^2/M^2$ . However note that according to a low-energy observer the value of  $\lambda$  just has to be taken from experiment.

More precise experiments could measure the first two terms in the expansion of the amplitude. These terms can be reproduced by the dimension-6 effective Lagrangian

$$\mathcal{L}_6 = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 - \frac{1}{4!} \lambda \phi^4 - \frac{1}{8} \lambda' \phi^2 \square \phi^2.$$

To reproduce (7.2) up to  $\mathcal{O}(1/M^2)$ , one has to set  $\lambda = -3g^2/M^2$  and  $\lambda' = g^2/M^4$ .

In this way one can construct a sequence of ever-more-accurate (but ever-more-complicated) effective Lagrangians  $\mathcal{L}_4, \mathcal{L}_6, \mathcal{L}_8, \dots$  that reproduce the first 1, 2, 3,  $\dots$  terms in the expansion of the scattering amplitude. In fact, in this simple theory, one can write down an “all-orders” effective Lagrangian for  $\phi$  that exactly reproduces all scattering amplitudes that only have external  $\phi$  particles:

$$\mathcal{L}_\infty = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 + \frac{1}{8} g^2 \phi^2 \frac{1}{\square + M^2} \phi^2. \quad (7.3)$$

Here you can regard the peculiar-looking  $1/(\square + M^2)$  as defined by the series

$$\frac{1}{\square + M^2} = \frac{1}{M^2} - \frac{\square}{M^4} + \frac{\square^2}{M^6} \mp \dots$$

But note that this whole effective field theory approach breaks down for scattering at energies  $E \sim M$ , when  $\chi$  particles can be produced.

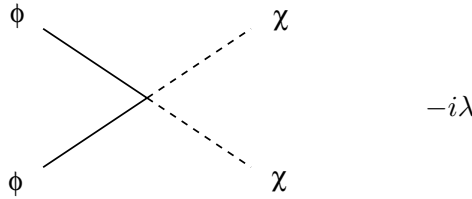
Moral of the story: think of  $\phi$  as describing observable physics at an energy scale  $E$ , while  $\chi$  describes some unknown high-energy physics at the scale  $M$ . You might think that  $\chi$  has no effect when  $E \ll M$ , but as we’ve seen, this just isn’t true. Rather high-energy physics leaves an imprint on low-energy phenomena, in a way that can be organized as an expansion in  $E/M$ . The leading behavior for  $E \ll M$  is captured by conventional  $\lambda\phi^4$  theory!

**7.1.2 Example II:  $\phi^2\chi^2$  theory**

As our next example consider

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 + \frac{1}{2}\partial_\mu\chi\partial^\mu\chi - \frac{1}{2}M^2\chi^2 - \frac{1}{4}\lambda\phi^2\chi^2$$

We'll continue to take  $m \ll M$ . So the only real change is that we now have a 4-point  $\phi^2\chi^2$  interaction; corresponding to this the coupling  $\lambda$  is dimensionless. The vertex is



As before we'll be interested in  $\phi$ - $\phi$  scattering at energies  $E \ll M$ . Given our previous example, at leading order we'd expect this to be described in terms of a dimension-4 effective Lagrangian

$$\mathcal{L}_{\text{eff}} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{1}{4!}\lambda_{\text{eff}}\phi^4$$

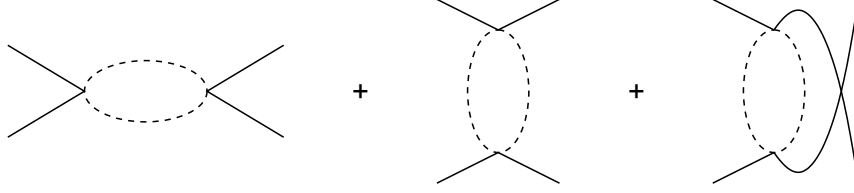
with some effective 4-point coupling  $\lambda_{\text{eff}}$ . Also we might expect that since  $\lambda_{\text{eff}}$  is dimensionless it can only depend on the dimensionless quantities in the problem, namely the underlying coupling and ratio of masses:  $\lambda_{\text{eff}} = \lambda_{\text{eff}}(\lambda, m/M)$ .

This argument turns out to be a bit too quick. To see what's actually going on let's do two computations, one in the effective theory and one in the underlying theory, and match the results. In the effective theory at leading order  $\phi$ - $\phi$  scattering is given by

$$\Rightarrow \quad -i\mathcal{M} = -i\lambda_{\text{eff}}$$

Here, just for simplicity, I've set the external momenta to zero.<sup>†</sup> On the other hand, in the underlying theory, the leading contribution to  $\phi$ - $\phi$  scattering comes from

<sup>†</sup> This means we aren't studying a physical scattering process. If this bothers you just imagine embedding this process inside a larger diagram. Alternatively, you can carry out the slightly more involved matching of on-shell scattering amplitudes.



$$\Rightarrow -i\mathcal{M} = 3 \cdot \frac{1}{2} \cdot (-i\lambda)^2 \int \frac{d^4k}{(2\pi)^4} \left( \frac{i}{k^2 - M^2} \right)^2$$

(the factor of three comes from the three diagrams, the factor of 1/2 is a symmetry factor – see Peskin & Schroeder p. 93). The two amplitudes agree provided

$$\lambda_{\text{eff}} = \frac{3i\lambda^2}{2} \int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2 - M^2)^2}.$$

The integral is, umm, divergent. We’ll fix this shortly by putting in a cutoff, but for now let’s just push on. The standard technique for doing loop integrals is to “Wick rotate” to Euclidean space. Define a Euclidean momentum

$$k_E^\mu = (-ik^0; \mathbf{k})$$

which satisfies

$$k_E^2 \equiv \delta_{\mu\nu} k_E^\mu k_E^\nu = -(k^0)^2 + |\mathbf{k}|^2 = -k^2 \equiv -g_{\mu\nu} k^\mu k^\nu.$$

By rotating the  $k^0$  contour of integration  $90^\circ$  counterclockwise in the complex plane<sup>†</sup> we can replace

$$\int_{-\infty}^{\infty} dk^0 \rightarrow \int_{-i\infty}^{i\infty} dk^0 = i \int_{-\infty}^{\infty} dk_E^0$$

to obtain

$$\lambda_{\text{eff}} = -\frac{3\lambda^2}{2} \int \frac{d^4k_E}{(2\pi)^4} \frac{1}{(k_E^2 + M^2)^2}.$$

The integrand is spherically symmetric so we can replace  $d^4k_E \rightarrow 2\pi^2 k_E^3 dk_E$  where  $2\pi^2$  is the “area” of a unit 3-sphere. So finally

$$\lambda_{\text{eff}} = -\frac{3\lambda^2}{16\pi^2} \int_0^\infty \frac{k_E^3 dk_E}{(k_E^2 + M^2)^2}.$$

<sup>†</sup> The integrand vanishes rapidly enough at large  $k^0$  to make this rotation possible. Also one has to mind the  $i\epsilon$ ’s. See Peskin & Schroeder p. 193.

To make sense of this we need some kind of cutoff, which you can think of as an ad-hoc, short-distance modification to the theory. A simple way to introduce a cutoff is to restrict  $|k_E| < \Lambda$ . We're left with

$$\lambda_{\text{eff}} = -\frac{3\lambda^2}{16\pi^2} \int_0^\Lambda \frac{k_E^3 dk_E}{(k_E^2 + M^2)^2} = -\frac{3\lambda^2}{32\pi^2} \left[ \log \frac{\Lambda^2 + M^2}{M^2} - \frac{\Lambda^2}{\Lambda^2 + M^2} \right]. \quad (7.4)$$

Moral of the story: you need a cutoff  $\Lambda$  to make sense of a quantum field theory. Low-energy physics can be described by an effective  $\phi^4$  theory with a coupling  $\lambda_{\text{eff}}$ . The value of  $\lambda_{\text{eff}}$  depends on the cutoff through the dimensionless ratio  $\Lambda/M$ .

### 7.1.3 Effective field theory generalities

The conventional wisdom on effective field theories:

- By “integrating out”<sup>†</sup> short-distance, high-energy degrees of freedom one can obtain an effective Lagrangian for the low energy degrees of freedom.
- The (all-orders) effective Lagrangian should contain *all possible* terms that are compatible with the symmetries of the underlying Lagrangian (even if those symmetries are spontaneously broken!). For example, in  $\phi^2\chi$  theory, the effective Lagrangian (7.3) respects the  $\phi \rightarrow -\phi$  symmetry of the underlying theory.
- The effective Lagrangian has to be respect dimensional analysis. However, in doing dimensional analysis, don't forget about the cutoff scale  $\Lambda$  of the underlying theory. For example, in  $\phi^2\chi^2$  theory, the effective dimensionless coupling (7.4) depends on the ratio  $\Lambda/M$ .

As an example of the power of this sort of reasoning, let's ask: what theory describes the massless Goldstone bosons associated with chiral symmetry breaking? The Goldstones can be described by a field

$$U(x) = e^{i\pi^a T^a / f} \in SU(3).$$

Terms in  $\mathcal{L}_{\text{eff}}$  with no derivatives are ruled out (remember  $U$  parametrizes the space of vacua, so the potential energy can't depend on  $U$ ). Terms with one derivative aren't Lorentz invariant. There's only one term with two derivatives that respects the symmetry  $U \rightarrow LUR^\dagger$ , so up to two derivatives

<sup>†</sup> The term comes from path integrals, where one does the functional integral over  $\chi$  first.

the effective action is

$$\mathcal{L}_{\text{eff}} = \frac{1}{4} f^2 \text{Tr} \left( \partial_\mu U^\dagger \partial^\mu U \right). \quad (7.5)$$

The coupling  $f$  has units of energy.<sup>†</sup> You can write down terms in the effective Lagrangian with more derivatives. But when you expand in powers of the Goldstone fields  $\pi^a$  such terms only contribute at dimension 6 and higher. So the low energy interactions of the Goldstone bosons, involving operators up to dimension 4, are completely fixed in terms of one undetermined parameter  $f$ .

As a further example of the power of effective field theory reasoning, recall the  $O(4)$  linear  $\sigma$ -model from the homework. This theory had an  $SO(4) \approx SU(2) \times SU(2)$  symmetry group which spontaneously broke to an  $SU(2)$  subgroup. Let's compare this to the behavior of QCD with two flavors of massless quarks. With two flavors QCD has an  $SU(2)_L \times SU(2)_R$  chiral symmetry that presumably spontaneously breaks to an  $SU(2)$  isospin subgroup. The symmetry breaking patterns are the same, so the low-energy dynamics of the Goldstone bosons are the same. This means that, from the point of view of a low energy observer, *QCD with two flavors of massless quarks cannot be distinguished from an  $O(4)$  linear  $\sigma$ -model*. Of course, to a high energy observer, the two theories could not be more different.

## 7.2 Renormalization

It's best to think of *all* quantum field theories as effective field theories. In particular one should always have a cutoff scale  $\Lambda$  in mind. It's important to recognize that this cutoff could arise in two different ways.

- (i) As a reflection of new short-distance physics (such as new types of particles or new types of interactions) that kick in at the scale  $\Lambda$ . In this case the cutoff is physical, in the sense that the theoretical framework really changes at the scale  $\Lambda$ .
- (ii) As a matter of convenience. It's very useful to focus on a certain energy scale – say set by the c.m. energy of a given scattering process – and ignore what's going on at much larger energy scales. To do this it's useful to put in a cutoff, even though it's not necessary in the sense that nothing special happens at the scale  $\Lambda$ .

<sup>†</sup> For pions, where  $U = e^{i\pi^a \sigma^a / f}$  is an  $SU(2)$  matrix, the coupling is denoted  $f_\pi$ . It's known as the pion decay constant for reasons you'll see in problem 8.2. Warning: the value for  $f_\pi$  is convention-dependent. With the normalizations we are using  $f_\pi = 93 \text{ MeV}$ .



Now that we have a cutoff in mind, an important question arises: how does the cutoff enter into physical quantities? This leads to the subject of renormalization. I'll illustrate it by way of a few examples.

### 7.2.1 Renormalization in $\phi^4$ theory

Suppose we have  $\phi^4$  theory with a cutoff  $\Lambda$  on the Euclidean loop momentum.

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{1}{4!}\lambda\phi^4$$

restrict  $|k_E| < \Lambda$

At face value we now have a three-dimensional space of theories labeled by the mass  $m$ , the value of the coupling  $\lambda$  and the value of the cutoff  $\Lambda$ . However some of these theories are equivalent as far as any low-energy observer can tell. We'd like to identify these families of equivalent theories. To find a particular family think of the parameters in our Lagrangian as depending on the value of the cutoff:  $m = m(\Lambda)$ ,  $\lambda = \lambda(\Lambda)$ . The functions  $m(\Lambda)$ ,  $\lambda(\Lambda)$  are determined by changing the cutoff and requiring that low-energy physics stays the same. For a preview of the results see figure 7.1.

To see how this works, suppose somebody decides to study  $\phi^4$  theory with a cutoff  $\Lambda$ .

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{1}{4!}\lambda\phi^4$$

restrict  $|k_E| < \Lambda$

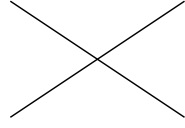
Someone else comes along and writes down an effective theory with a smaller cutoff,  $\Lambda' = \Lambda - \delta\Lambda$ . Denoting this theory with primes

$$\mathcal{L}' = \frac{1}{2}\partial_\mu\phi'\partial^\mu\phi' - \frac{1}{2}m'^2\phi'^2 - \frac{1}{4!}\lambda'\phi'^4$$

restrict  $|k_E| < \Lambda'$

These theories will be equivalent at low energies provided we relate the parameters in an appropriate way. To relate  $m$  and  $m'$  we require that the  $\phi$  and  $\phi'$  propagators have the same behavior at low energy: you'll see this on the homework, so I won't go into details here. To relate  $\lambda$  and  $\lambda'$  we require that low-energy scattering amplitudes agree.

With this motivation let's study  $\phi$ - $\phi$  scattering at zero momentum. At tree level in the primed theory we have


 $\Rightarrow -i\mathcal{M}' = -i\lambda'$

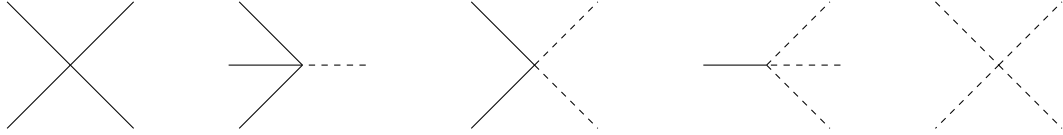
In the unprimed theory let's decompose  $\phi = \phi' + \chi$ , where the Euclidean loop momenta of these fields are restricted so that

$$\begin{aligned} \phi & \text{ includes all Fourier modes with } |k_E| < \Lambda \\ \phi' & \text{ includes all modes with } |k_E| < \Lambda' \\ \chi & \text{ only has modes with } \Lambda' < |k_E| < \Lambda \end{aligned} \quad (7.6)$$

We get some complicated-looking interactions

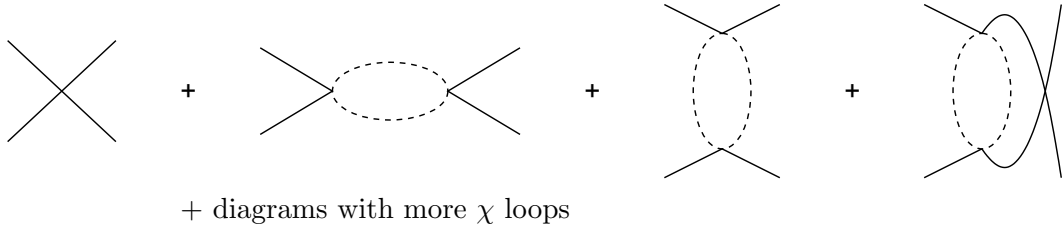
$$\mathcal{L}_{\text{int}} = -\frac{1}{4!}\lambda\phi^4 = -\frac{1}{4!}\lambda(\phi'^4 + 4\phi'^3\chi + 6\phi'^2\chi^2 + 4\phi'\chi^3 + \chi^4)$$

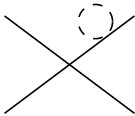
corresponding to vertices



Here a solid line represents a  $\phi'$  particle, while a dashed line represents a  $\chi$  particle; the Feynman rules for all these interactions are the same: just  $-i\lambda$ .

In the primed theory we did a tree-level calculation. In the unprimed theory this corresponds to a calculation where we have no  $\phi'$  loops but arbitrary numbers of  $\chi$  loops:



(we're neglecting diagrams such as  that get absorbed into the relationship between  $m$  and  $m'$ ). I'll stop at a single  $\chi$  loop. We encountered

these diagrams in our  $\phi^2\chi^2$  example, so we can write down the answer immediately.

$$-i\mathcal{M} = -i\lambda + \frac{3i\lambda^2}{2} \int_{\Lambda'}^{\Lambda} \frac{d^4k_E}{(2\pi)^4} \frac{1}{(k_E^2 + m^2)^2}$$

Note that the range of  $\chi$  momenta is restricted according to (7.6).

Our two effective field theories will agree provided  $\mathcal{M}' = \mathcal{M}$  or equivalently

$$\lambda' = \lambda - \frac{3\lambda^2}{2} \int_{\Lambda'}^{\Lambda} \frac{d^4k_E}{(2\pi)^4} \frac{1}{(k_E^2 + m^2)^2}$$

Just to simplify things let's assume  $\Lambda \gg m$  so that

$$\lambda' = \lambda - \frac{3\lambda^2}{16\pi^2} \int_{\Lambda'}^{\Lambda} \frac{dk_E}{k_E}.$$

If we take  $\delta\Lambda = \Lambda - \Lambda'$  to be infinitesimal we get

$$\lambda - \frac{d\lambda}{d\Lambda} \delta\Lambda = \lambda - \frac{3\lambda^2}{16\pi^2} \frac{\delta\Lambda}{\Lambda}.$$

Thus we've obtained a differential equation that determines the cutoff dependence of the coupling.

$$\frac{d\lambda}{d\Lambda} = \frac{3\lambda^2}{16\pi^2\Lambda} \quad \Rightarrow \quad \frac{1}{\lambda(\Lambda)} = -\frac{3}{16\pi^2} \log \Lambda + \text{const.}$$

It's convenient to specify the constant of integration by choosing an arbitrary energy scale  $\mu$  and writing

$$\boxed{\frac{1}{\lambda(\Lambda)} = \frac{1}{\lambda(\mu)} - \frac{3}{16\pi^2} \log \frac{\Lambda}{\mu}} \quad (7.7)$$

Here  $\mu$  is the “renormalization scale” and  $\lambda(\mu)$  is the one-loop “running coupling” or “renormalized coupling”.

Buzzwords: for each value of  $\lambda(\mu)$  we've found a family of effective field theories, related by “renormalization group flow,” whose physical consequences at low energies are the same. Each family makes up a curve or “renormalization group trajectory” in the  $(\lambda, \Lambda)$  plane. This is illustrated in figure 7.1. The scale dependence of the coupling is controlled by the “ $\beta$ -function”

$$\beta(\lambda) \equiv \frac{d\lambda}{d \log \Lambda} = \frac{3\lambda^2}{16\pi^2} + \mathcal{O}(\lambda^3)$$

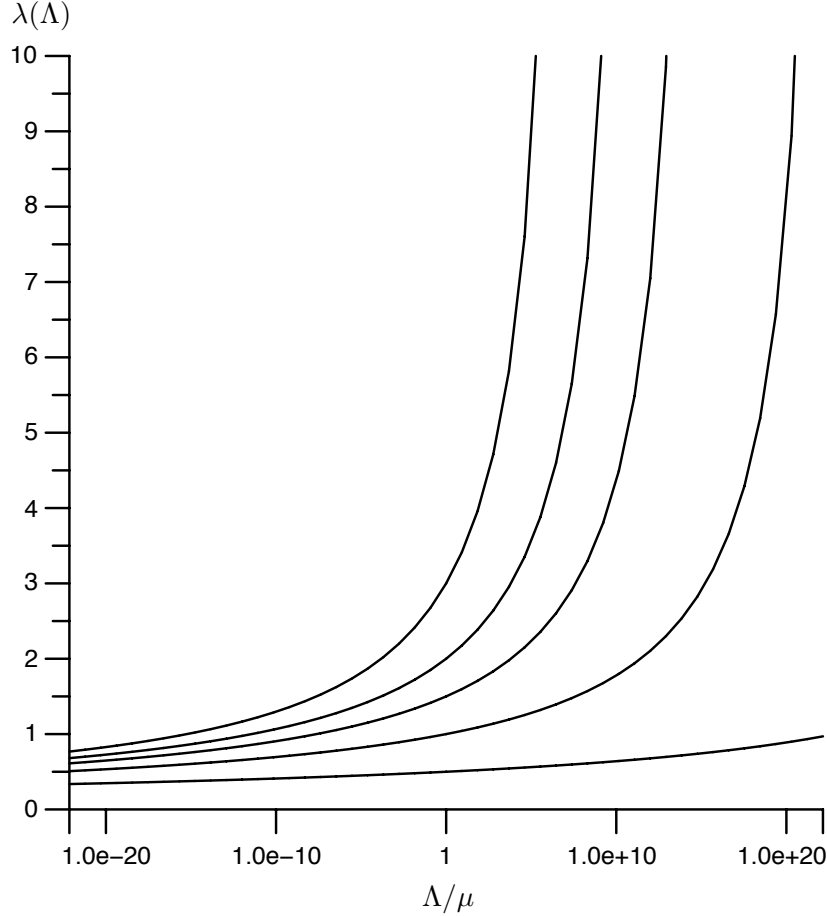


Fig. 7.1. One-loop running coupling versus scale for  $\lambda(\mu) = 0.5, 1.0, 1.5, 2.0, 3.0$ . Each curve represents a single renormalization group trajectory. Note that the horizontal axis is on a log scale.

The solution to this differential equation, given in (7.7), is absolutely fundamental: it tells you how the coupling has to be changed in order to compensate for a change in cutoff. Equivalently, if you regard  $\Lambda$  and the bare coupling  $\lambda(\Lambda)$  as fixed, it tells you how the *renormalized* coupling has to be changed if you shift your renormalization scale  $\mu$ .

According to (7.7) the bare coupling vanishes as  $\Lambda \rightarrow 0$ . As  $\Lambda$  increases the degrees of freedom at the cutoff scale become more and more strongly coupled. In fact, if you take (7.7) seriously, the bare coupling diverges at  $\Lambda = \Lambda_{\text{max}} = \mu e^{16\pi^2/3\lambda(\mu)}$ . Of course our perturbative analysis isn't trustworthy once the theory becomes strongly coupled. But we can reach an interesting

conclusion: *something* has to happen before the scale  $\Lambda_{\text{max}}$ . At the very least perturbation theory has to break down.†

### 7.2.2 Renormalization in QED

As a second example let's look at renormalization in QED. We'll concentrate on the so-called “field strength renormalization” of the electromagnetic field, since this turns out to be responsible for the running of electric charge.

Consider QED with a cutoff  $\Lambda$  on the Euclidean loop momentum.

$$\mathcal{L} = -\frac{1}{4}\xi F_{\mu\nu}F^{\mu\nu} + \bar{\psi} [i\gamma^\mu(\partial_\mu + ieQA_\mu) - m] \psi$$

restrict  $|k_E| < \Lambda$

Here we've generalized the QED Lagrangian slightly, by introducing an arbitrary normalization constant  $\xi$  in front of the Maxwell kinetic term. If we lower the cutoff a bit, to  $\Lambda' = \Lambda - \delta\Lambda$ , we'd write down a new theory†

$$\mathcal{L}' = -\frac{1}{4}\xi' F_{\mu\nu}F^{\mu\nu} + \bar{\psi} [i\gamma^\mu(\partial_\mu + ie'QA_\mu) - m'] \psi$$

restrict  $|k_E| < \Lambda'$

The two theories will agree provided we relate  $\xi$  and  $\xi'$  appropriately. We'll neglect the differences between  $e, m$  and  $e', m'$  since it turns out they don't matter for our purposes. Likewise we'll neglect the possibility of putting a normalization constant in front of the Dirac kinetic term.

To fix the relation between  $\xi$  and  $\xi'$  we require that the photon propagators computed in the two theories agree at low energy. Rather than match propagators directly, it's a bit simpler to use

$$\xi' \equiv \xi(\Lambda') = \xi(\Lambda) - \frac{d\xi}{d\Lambda} \delta\Lambda.$$

Plugging this into the primed Lagrangian and comparing the two theories, the primed Lagrangian has an extra term

$$\frac{1}{4} \frac{d\xi}{d\Lambda} \delta\Lambda F_{\mu\nu}F^{\mu\nu} = -\frac{1}{2} \frac{d\xi}{d\Lambda} \delta\Lambda A^\mu \left( g_{\mu\nu} \partial_\lambda \partial^\lambda - \partial_\mu \partial_\nu \right) A^\nu$$

which corresponds to a two-photon vertex

† Something more dramatic probably has to happen. It's likely that one can't make sense of the theory when  $\Lambda$  gets too large. See Weinberg, QFT vol. II p. 137.

† I've gotten lazy and haven't bothered putting primes on the fields in  $\mathcal{L}'$ ; it should be clear from the context what range of Euclidean momenta is allowed.

$$\mu \text{ --- } \overbrace{\text{wavy line}}^{\text{k}} \text{ --- } \text{v} \quad i \frac{d\xi}{d\Lambda} \delta\Lambda (g_{\mu\nu} k^2 - k_\mu k_\nu)$$

In the unprimed theory, on the other hand, the photon propagator receives corrections from the vacuum polarization diagram studied in appendix C.

In order for the two theories to agree we must have

$$\mu \text{ --- } \overbrace{\text{wavy line}}^{\text{k}} \text{ --- } \text{v} = \mu \text{ --- } \overbrace{\text{wavy line}}^{\text{k}} \text{ --- } \text{circle} \text{ --- } \overbrace{\text{wavy line}}^{\text{k}} \text{ --- } \text{v}$$

In equations this means

$$i \frac{d\xi}{d\Lambda} \delta\Lambda (g_{\mu\nu} k^2 - k_\mu k_\nu) = -4e^2 Q^2 (g_{\mu\nu} k^2 - k_\mu k_\nu) \int_0^1 dx \int \frac{d^4 q}{(2\pi)^4} \frac{2x(1-x)}{(q^2 + k^2 x(1-x) - m^2)^2}$$

where the electron loop momentum is restricted to  $\Lambda' < |q_E| < \Lambda$ . Note that the projection operators  $g_{\mu\nu} k^2 - k_\mu k_\nu$  cancel. Let's do the matching at  $k^2 = 0$ , and for simplicity let's neglect the electron mass relative to the cutoff. Then Wick rotating we get

$$i \frac{d\xi}{d\Lambda} \delta\Lambda = -4e^2 Q^2 \underbrace{\int_0^1 dx 2x(1-x)}_{=1/3} \underbrace{\int_{\Lambda'}^{\Lambda} \frac{id^4 q_E}{(2\pi)^4} \frac{1}{q_E^4}}_{=i\delta\Lambda/8\pi^2\Lambda}$$

or

$$\frac{d\xi}{d\Lambda} = -\frac{e^2 Q^2}{6\pi^2\Lambda}.$$

This means the normalization of the Maxwell kinetic term depends on the cutoff. To see the physical significance of this fact it's useful to rescale the gauge field  $A_\mu \rightarrow \frac{1}{\sqrt{\xi}} A_\mu$ . The rescaled gauge field has a canonical kinetic term. However from the form of the covariant derivative  $\mathcal{D}_\mu = \partial_\mu + ieQA_\mu$  we see that the physical electric charge – the quantity that shows up in the vertex for emitting a canonically-normalized photon – is given by  $e_{\text{phys}} =$

$e/\sqrt{\xi}$ . This evolves with scale according to<sup>†</sup>

$$\frac{d}{d\Lambda} e_{\text{phys}}^2 = \frac{d}{d\Lambda} \frac{e^2}{\xi} = -\frac{e^2}{\xi^2} \frac{d\xi}{d\Lambda} = \frac{e_{\text{phys}}^4 Q^2}{6\pi^2 \Lambda}.$$

Introducing an arbitrary renormalization scale  $\mu$ , we can write the solution to this differential equation as

$$\frac{1}{e_{\text{phys}}^2(\Lambda)} = \frac{1}{e_{\text{phys}}^2(\mu)} - \frac{Q^2}{6\pi^2} \log \frac{\Lambda}{\mu}.$$

Qualitatively the behavior of QED is pretty similar to  $\phi^4$  theory. Neglecting the electron mass the physical electric charge goes to zero at long distances, while at short distances QED becomes more and more strongly coupled. In the one-loop approximation the physical electric charge blows up when  $\Lambda = \Lambda_{\text{max}} = \mu e^{6\pi^2/e_{\text{phys}}^2(\mu)Q^2}$ .

### 7.2.3 Comments on renormalization

The sort of analysis we have done is very powerful. The renormalization group packages the way in which the cutoff can enter in physical quantities. By reorganizing perturbation theory as an expansion in powers of the renormalized coupling  $\lambda(\mu)$  rather than the bare coupling  $\lambda(\Lambda)$  one can express scattering amplitudes in terms of finite measurable quantities. (“Finite” in the sense that  $\lambda(\mu)$  is independent of the cutoff, and “measurable” in the sense that  $\lambda(\mu)$  can be extracted from experimental input – say the cross section for  $\phi - \phi$  scattering measured at some energy scale.)

That’s how renormalization was first introduced: as a tool for handling divergent Feynman diagrams. But renormalization is not merely a technique for understanding cutoff dependence. The relationship between  $\lambda(\mu)$  and  $\lambda(\Lambda)$  is non-linear. This means renormalization mixes different orders in perturbation theory. By choosing  $\mu$  appropriately one can “improve” the reorganized perturbation theory (that is, make the leading term as dominant as possible). You’ll see examples of this on the homework.

Finally let me comment on the relation between Wilson’s approach to renormalization as described here and the more conventional field theory approach. In the conventional approach one always has the  $\Lambda \rightarrow \infty$  limit in mind. The Lagrangian at the scale  $\Lambda$  is referred to as the “bare Lagrangian,”

<sup>†</sup> We’re assuming the parameter  $e^2$  doesn’t depend on  $\Lambda$ . To establish this one has to do some further analysis: Peskin and Schroeder p. 334 or Ramond p. 256. Alternatively one can bypass this issue by working with external static charges as in problem 7.4.

while the Lagrangian at the scale  $\mu$  is the “renormalized Lagrangian.” One builds up the difference between  $\mathcal{L}(\Lambda)$  and  $\mathcal{L}(\mu)$  order-by-order in perturbation theory, by adding “counterterms” to the renormalized Lagrangian. In doing this one holds the renormalized couplings  $\lambda(\mu)$  fixed by imposing “renormalization conditions” on scattering amplitudes.

## References

EFFECTIVE FIELD THEORY. There are some excellent review articles on effective field theory: see D. B. Kaplan, *Effective field theories*, arXiv:nucl-th/9506035 or A. Manohar, *Effective field theories*, arXiv:hep-ph/9606222.

RENORMALIZATION. A discussion of Wilson’s approach to renormalization can be found in chapter 12.1 of Peskin & Schroeder.

CHIRAL PERTURBATION THEORY. We’ve discussed effective Lagrangians, which provide a systematic way of describing the dynamics of a theory at low energies. Given an effective Lagrangian one can compute low-energy scattering amplitudes perturbatively, as an expansion in powers of the momentum of the external lines. For strong interactions this technique is known as chiral perturbation theory. The pion scattering of problem 7.1 is an example of lowest-order  $\chi$ PT. For a review see Bastian Kubis, *An introduction to chiral perturbation theory*, arXiv:hep-ph/0703274.

SCALE OF CHIRAL PERTURBATION THEORY. Chiral perturbation theory provides a systematic low-energy approximation for computing scattering amplitudes. But we should ask: low energy relative to what? To get a handle on this, note that in the effective Lagrangian for pions (7.5)  $1/f_\pi^2$  acts as a loop-counting parameter (analogous to  $\lambda$  in  $\phi^4$  theory, or  $e^2$  in QED). Moreover each loop is usually associated with a numerical factor  $1/16\pi^2$ , as explained on p. 102. So the loop expansion is really an expansion in powers of  $(\text{energy})^2/(4\pi f_\pi)^2$ . It’s usually assumed that higher dimension terms in the pion effective Lagrangian will be suppressed by powers of the same scale, namely  $4\pi f_\pi \sim 1 \text{ GeV}$ . See Manohar and Georgi, *Nucl. Phys.* **B234** (1984) 189.

## Exercises

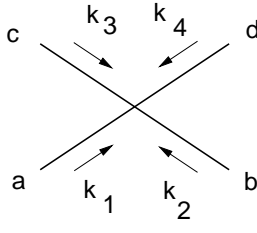
### 7.1 $\pi - \pi$ scattering



If you take the pion effective Lagrangian

$$\mathcal{L} = \frac{1}{4}f_\pi^2 \text{Tr}(\partial_\mu U^\dagger \partial^\mu U) + 2\mu^3 \text{Tr}(M(U + U^\dagger))$$

and expand it to fourth order in the pion fields you find interaction terms that describe low-energy  $\pi - \pi$  scattering. Conventions:  $U = e^{i\pi^a \sigma^a / f_\pi}$  is an  $SU(2)$  matrix where  $\sigma^a$  are Pauli matrices and  $f_\pi = 93 \text{ MeV}$ . I'm working in a Cartesian basis where  $a = 1, 2, 3$ . For simplicity I'll set  $m_u = m_d$  so that isospin is an exact symmetry. The resulting 4-pion vertex is (sorry)



$$-\frac{i}{3f_\pi^2} \left[ \delta^{ab} \delta^{cd} \left( (k_1 + k_2) \cdot (k_3 + k_4) - 2k_1 \cdot k_2 - 2k_3 \cdot k_4 - m_\pi^2 \right) \right. \\ \left. + \delta^{ac} \delta^{bd} \left( (k_1 + k_3) \cdot (k_2 + k_4) - 2k_1 \cdot k_3 - 2k_2 \cdot k_4 - m_\pi^2 \right) \right. \\ \left. + \delta^{ad} \delta^{bc} \left( (k_1 + k_4) \cdot (k_2 + k_3) - 2k_1 \cdot k_4 - 2k_2 \cdot k_3 - m_\pi^2 \right) \right]$$

Note that all momenta are directed inwards in the vertex.

(i) Compute the scattering amplitude for  $\pi^a(k_1) \pi^b(k_2) \rightarrow \pi^c(k_3) \pi^d(k_4)$ . Here  $a, b, c, d$  are isospin labels and  $k_1, k_2, k_3, k_4$  are external momenta.

(ii) Since we have two pions the initial state could have total isospin  $I = 0, 1, 2$ . Extract the scattering amplitude in the various isospin channels by putting in initial isospin wavefunctions proportional to

$$I = 0 : \delta^{ab} \quad I = 1 : (\text{antisymmetric})^{ab} \quad I = 2 : (\text{symmetric traceless})^{ab}$$

(iii) Evaluate the amplitudes in the various channels “at threshold” (meaning in the limit where the pions have vanishing spatial momentum).

(iv) Threshold scattering amplitudes are usually expressed in terms of “scattering lengths” defined (for  $s$ -wave scattering) by  $a = -\mathcal{M}/32\pi m_\pi$ .<sup>†</sup> For  $I = 0, 2$  the experimental values and statistical errors are (Brookhaven E865 collaboration, arXiv:hep-ex/0301040)

$$a_{I=0} = (0.216 \pm 0.013)m_\pi^{-1} \quad a_{I=2} = (-0.0454 \pm 0.0031)m_\pi^{-1}$$

How well did you do?

<sup>†</sup> This is in the convention where the sum of Feynman diagrams gives  $-i\mathcal{M}$ . For a complete discussion of pion scattering see section VI-4 in Donoghue et. al., *Dynamics of the standard model*.

### 7.2 Mass renormalization in $\phi^4$ theory

Let's study renormalization of the mass parameter in  $\phi^4$  theory. As in the notes we consider

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 - \frac{1}{4!} \lambda \phi^4$$

with a cutoff on the Euclidean loop momentum  $\Lambda$ , and

$$\mathcal{L}' = \frac{1}{2} \partial_\mu \phi' \partial^\mu \phi' - \frac{1}{2} m'^2 \phi'^2 - \frac{1}{4!} \lambda' \phi'^4$$

with a cutoff  $\Lambda' = \Lambda - \delta\Lambda$ . The idea is to match the tree-level propagator in the primed theory to the corresponding quantity in the unprimed theory, namely the sum of diagrams



(i) In the primed theory the propagator is

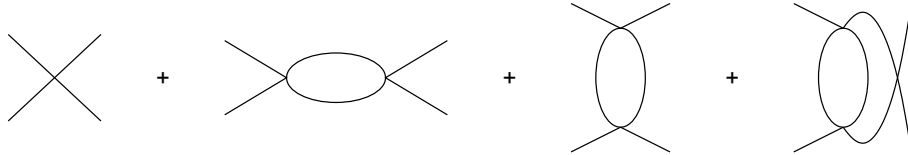
$$\frac{i}{p^2 - m'^2}.$$

Set  $m'^2 = m^2 + \delta m^2$  where  $\delta m^2 = -\frac{dm^2}{d\Lambda} \delta\Lambda$ . Expand the propagator to first order in  $\delta\Lambda$ .

(ii) Match your answer to the corresponding calculation in the unprimed theory. You can stop at a single  $\chi$  loop, and for simplicity you can assume  $m \ll \Lambda$ . You should obtain a trivial differential equation for the mass parameter and solve it to find  $m^2(\Lambda)$ .

### 7.3 Renormalization and scattering

(i) Write down the one-loop four-point scattering amplitude in  $\frac{1}{4!} \lambda \phi^4$  theory, coming from the diagrams



To regulate the diagrams you should Wick rotate to Euclidean space and put a cutoff on the magnitude of the Euclidean momentum:  $k_{\text{Euclidean}}^2 < \Lambda^2$ . You should keep the external momenta

non-zero, however you don't need to work at evaluating any loop integrals.

- (ii) It's useful to reorganize perturbation theory as an expansion in the renormalized coupling  $\lambda(\mu)$ , defined by

$$\frac{1}{\lambda(\Lambda)} = \frac{1}{\lambda(\mu)} - \frac{3}{16\pi^2} \log \frac{\Lambda}{\mu}.$$

Rewrite your scattering amplitude as an expansion in powers of  $\lambda(\mu)$  up to  $\mathcal{O}(\lambda(\mu)^2)$ .

- (iii) You can “improve” perturbation theory by choosing  $\mu$  in order to make the  $\mathcal{O}(\lambda(\mu)^2)$  terms in your scattering amplitude as small as possible. Suppose you were interested in soft scattering,  $s \approx t \approx u \approx 0$ . What value of  $\mu$  should you use? Alternatively, suppose you were interested in the “deep Euclidean” regime where  $s, t, u$  are large and negative (meaning  $s \approx t \approx u \ll -m^2$ ). Now what value of  $\mu$  should you use? (Here  $s, t, u$  are the usual Mandelstam variables. The values I'm suggesting do not satisfy the mass-shell condition  $s + t + u = 4m^2$ ; if this bothers you imagine embedding the four-point amplitude inside a larger diagram.)

Moral of the story: it's best to work in terms of a renormalized coupling evaluated at the energy scale relevant to the process you're considering.

#### 7.4 Renormalized Coulomb potential

Consider coupling the electromagnetic field to a conserved external current  $J^\mu(x)$ . The Lagrangian is

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - J_\mu A^\mu.$$

The Feynman rules for this theory are

$$\begin{array}{c} \text{Diagram 1: } \mu \text{ --- } \overbrace{\text{wavy line}}^k \text{ --- } \nu \quad \rightarrow \quad \frac{-ig_{\mu\nu}}{k^2} \\ \text{Diagram 2: } \text{X} \text{ --- } \overbrace{\text{wavy line}}^k \text{ --- } \mu \quad \rightarrow \quad -iJ_\mu(k) \end{array}$$

where  $J_\mu(k) = \int d^4x e^{ik \cdot x} J_\mu(x)$ . These rules are set up so the sum of connected Feynman diagrams gives  $-i \int d^4x \mathcal{H}_{\text{int}}$  where  $\mathcal{H}_{\text{int}}$  is the energy density due to interactions.

- (i) Introduce two point charges  $Q_1, Q_2$  at positions  $\mathbf{x}_1, \mathbf{x}_2$  by setting

$$J^0(x) = eQ_1\delta^3(\mathbf{x} - \mathbf{x}_1) + eQ_2\delta^3(\mathbf{x} - \mathbf{x}_2)$$

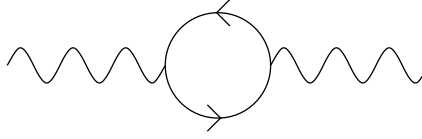
$$J^i = 0 \quad i = 1, 2, 3$$

We're measuring the charges in units of  $e = \sqrt{4\pi\alpha}$ . Compute the interaction energy by evaluating the diagram



You should integrate over the photon momentum. Do you recover the usual Coulomb potential?

- (ii) The photon propagator receives corrections from a virtual  $e^+ - e^-$  loop via the diagram



As shown in appendix C, this diagram equals

$$-4e^2 (g_{\mu\nu}k^2 - k_\mu k_\nu) \int_0^1 dx \int_{|q_E| < \Lambda} \frac{d^4q}{(2\pi)^4} \frac{2x(1-x)}{(q^2 + k^2x(1-x) - m^2)^2}. \quad (7.8)$$

Here  $m$  is the electron mass and  $\Lambda$  is a cutoff on the Euclidean loop momentum. Note that we haven't included photon propagators on the external lines in (7.8). Use this result to write an expression for the tree-level plus one-loop potential between two static charges. There's no need to evaluate any integrals at this stage.

- (iii) Use your result in part (ii) to derive the running coupling constant as follows. Set the electron mass to zero for simplicity. Consider changing the value of the cutoff,  $\Lambda \rightarrow \Lambda - \delta\Lambda$ . Allow the electric charge to depend on  $\Lambda$ ,  $e^2 \rightarrow e^2(\Lambda)$ , and show that up to order  $e^4$  the tree plus one-loop potential between two widely separated charges is independent of  $\Lambda$  provided

$$\frac{de^2}{d\Lambda} = \frac{e^4}{6\pi^2\Lambda}$$

or equivalently

$$\frac{1}{e^2(\Lambda)} = \frac{1}{e^2(\mu)} - \frac{1}{6\pi^2} \log \frac{\Lambda}{\mu}.$$

Here  $\mu$  is an arbitrary renormalization scale. Hints: for widely separated charges the typical photon momentum  $k$  is negligible compared to the cutoff  $\Lambda$ . Also since the external current is conserved,  $k_\mu J^\mu(k) = 0$ , you can drop corrections to the photon propagator proportional to  $k^\mu$ .

- (iv) Similar to problem 7.3: suppose you were interested in the potential between two unit charges separated by a distance  $r$ . You can still set the electron mass to zero. Working in terms of the renormalized coupling the tree diagram gives a potential  $e^2(\mu)/4\pi r$ . How should you choose the renormalization scale to make the loop corrections to this as small as possible? Hint: think about which photon momentum makes the dominant contribution to the potential.
- (v) Re-do part (iii), but keeping track of the electron mass. It's convenient to set the renormalization scale  $\mu$  to zero, that is, to solve for  $e^2(\Lambda)$  in terms of  $e^2(0)$ . Expand your answer to find how  $e^2(\Lambda)$  behaves for  $\Lambda \gg m$  and for  $\Lambda \ll m$ . Make a qualitative sketch of  $e^2(\Lambda)$ .

Moral of the story: matching the potential between widely separated charges provides a way to obtain the physical running coupling in QED. As always, you should choose  $\mu$  to reflect the important energy scale in the problem. Finally the electron mass cuts off the running of the coupling, which is why we're used to thinking of  $e$  as a fixed constant!

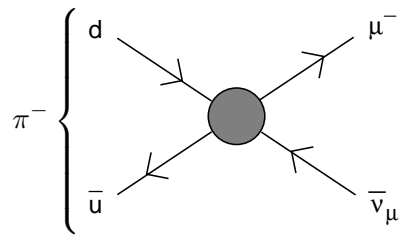
# 8

## Effective weak interactions: 4-Fermi theory

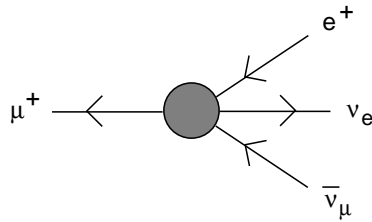
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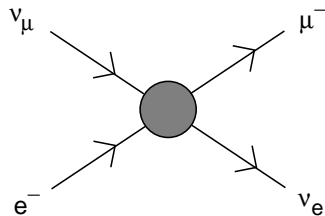
A typical weak process is pion decay  $\pi^- \rightarrow \mu^- \bar{\nu}_\mu$ .



Another typical process is muon decay  $\mu^+ \rightarrow e^+ \nu_e \bar{\nu}_\mu$ .



This is closely related to ‘inverse muon decay,’ or scattering  $\nu_\mu e^- \rightarrow \mu^- \nu_e$ .



The amplitudes for muon and inverse muon decay are related by crossing

symmetry. What's nice about IMD is that it's experimentally accessible (you can make  $\nu_\mu$  beams by letting pions decay in-flight).

We'd like to write an effective Lagrangian which can describe these sorts of weak interactions at low energies. Unfortunately the general principles of effective field theory don't get us very far. For example, to describe muon or inverse muon decay, they'd tell us to write down the most general Lorentz-invariant coupling of four spinor fields  $\mu$ ,  $\nu_\mu$ ,  $e$ ,  $\nu_e$  (the names of the particles stand for the corresponding Dirac fields). The problem is there are many such couplings. Fortunately Fermi (in 1934!) proposed a much more predictive theory which, with some parity-violating modifications, turned out to be right.

With no further ado, the effective Lagrangian which describes muon or inverse muon decay is

$$\begin{aligned}\mathcal{L}_1 &= -\frac{1}{\sqrt{2}}G_F [\bar{\mu}\gamma^\alpha(1-\gamma^5)\nu_\mu \bar{\nu}_e\gamma_\alpha(1-\gamma^5)e + \text{c.c.}] \\ \text{c.c.} &= \bar{\nu}_\mu\gamma^\alpha(1-\gamma^5)\mu \bar{e}\gamma_\alpha(1-\gamma^5)\nu_e\end{aligned}\quad (8.1)$$

Here Fermi's constant  $G_F = 1.2 \times 10^{-5} \text{ GeV}^{-2}$ . A very similar-looking interaction describes pion decay, namely

$$\mathcal{L}_2 = -\frac{1}{\sqrt{2}}G_F \cos \theta_C [\bar{\mu}\gamma^\alpha(1-\gamma^5)\nu_\mu \bar{u}\gamma_\alpha(1-\gamma^5)d + \text{c.c.}]$$

The only difference in structure between  $\mathcal{L}_1$  and  $\mathcal{L}_2$  is that the Cabibbo angle  $\theta_C = 13^\circ$  reflects quark mixing, a subject we'll say more about later. One can write down similar 4-Fermi interactions for other weak processes. A crucial feature of all these Lagrangians is that†

weak interactions only couple to left-handed chiral spinors

 (8.2)

To see this recall that  $P_L = \frac{1}{2}(1-\gamma^5)$  is a left-handed projection operator. In terms of  $\psi_L \equiv P_L\psi$ ,  $\bar{\psi}_L = (\psi_L)^\dagger\gamma^0$

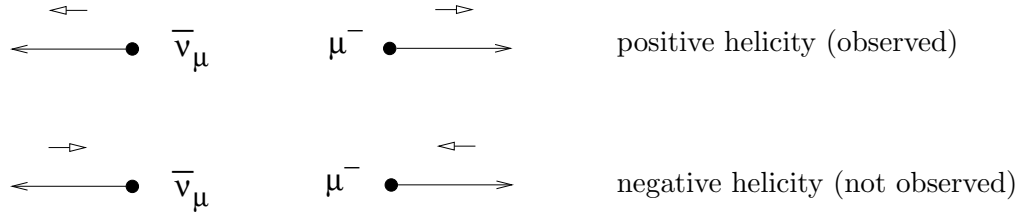
$$\mathcal{L}_1 = -2\sqrt{2}G_F [\bar{\mu}_L\gamma^\alpha\nu_{\mu L} \bar{\nu}_{e L}\gamma_\alpha e_L + \text{c.c.}]$$

which makes it clear that only left-handed spinors enter. This is often referred to as the “ $V-A$ ” structure of weak interactions (for “vector minus axial vector”).

Observational evidence for  $V-A$  comes from the decay  $\pi^- \rightarrow \mu^- \bar{\nu}_\mu$ . The pion is spinless. In the center of mass frame the muon and antineutrino

† more precisely this holds for “charged current” weak interactions. We'll get to weak neutral currents later.

come out back-to-back, with no orbital angular momentum along their direction of motion. So just from conservation of angular momentum there are two possible final state polarizations: both particles right-handed (positive helicity) or both particles left-handed (negative helicity).



It's found experimentally that only right-handed muons and antineutrinos are produced. This seemingly minor fact has far-reaching consequences.

- (i) Momentum is a vector and angular momentum is a pseudovector, so the two final states pictured above are exchanged by parity (plus a  $180^\circ$  spatial rotation). The fact that only one final state is observed means that weak interactions violate parity, and in fact violate it maximally.
- (ii) For a massless particle such as an antineutrino helicity and chirality are related. A right-handed antineutrino sits inside a left-handed chiral spinor, as required to participate in weak interactions according to (8.2).
- (iii) Wait a minute, you say, what about the muon? If the muon were massless then a right-handed muon would sit in a right-handed chiral spinor and shouldn't participate in weak interactions. It's the non-zero muon mass that breaks the connection between helicity and chirality and allows the muon to come out with the "wrong" polarization.
- (iv) This leads to an interesting prediction: in the limit of vanishing muon mass the decay  $\pi^- \rightarrow \mu^- \bar{\nu}_\mu$  is forbidden. Of course we can't change the muon mass. But we can compare the rates for  $\pi^- \rightarrow \mu^- \bar{\nu}_\mu$  and  $\pi^- \rightarrow e^- \bar{\nu}_e$ . The branching ratios are

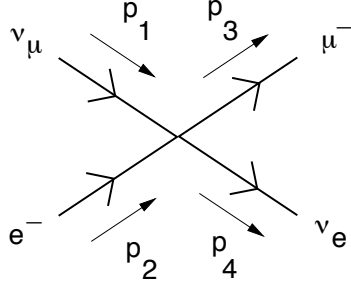
$$\text{B.R.}(\pi^- \rightarrow \mu^- \bar{\nu}_\mu) \approx 1$$

$$\text{B.R.}(\pi^- \rightarrow e^- \bar{\nu}_e) = 1.23 \times 10^{-4}$$

Pions prefer to decay to muons, even though phase space favors electrons as a decay product!



Having given some evidence for the form of the weak interaction Lagrangian let's calculate the amplitude for inverse muon decay.



$$-i\mathcal{M} = -\frac{i}{\sqrt{2}}G_F \bar{u}(p_3)\gamma_\alpha(1-\gamma^5)u(p_1) \bar{u}(p_4)\gamma_\alpha(1-\gamma^5)u(p_2) \quad (8.3)$$

$$\sum_{\text{spins}} |\mathcal{M}|^2 = \frac{1}{2}G_F^2 \text{Tr} \left( (\not{p}_3 + m_\mu)\gamma^\alpha(1-\gamma^5)\not{p}_1(1+\gamma^5)\gamma^\beta \right) \\ \text{Tr} \left( \not{p}_4\gamma_\alpha(1-\gamma^5)(\not{p}_2 + m_e)(1+\gamma^5)\gamma_\beta \right)$$

The electron and muon masses drop out since the trace of an odd number of Dirac matrices vanishes. Also the chiral projection operators can be combined to give

$$\sum_{\text{spins}} |\mathcal{M}|^2 = 2G_F^2 \text{Tr} \left( \not{p}_3\gamma^\alpha\not{p}_1(1+\gamma^5)\gamma^\beta \right) \text{Tr} \left( \not{p}_4\gamma_\alpha\not{p}_2(1+\gamma^5)\gamma_\beta \right)$$

You just have to grind through the remaining traces; for details see Quigg p. 90. The result is quite simple,

$$\sum_{\text{spins}} |\mathcal{M}|^2 = 128G_F^2 p_1 \cdot p_2 p_3 \cdot p_4 \quad (8.4)$$

Dividing by two to average over the electron spin gives  $\langle |\mathcal{M}|^2 \rangle = 64G_F^2 p_1 \cdot p_2 p_3 \cdot p_4$  (the  $\nu_\mu$ 's are polarized so we don't need to average over their spin). At high energies we can neglect the electron and muon masses and take

$$\left( \frac{d\sigma}{d\Omega} \right)_{\text{cm}} = \frac{\langle |\mathcal{M}|^2 \rangle}{64\pi^2 s} = \frac{G_F^2 s}{4\pi^2} \\ \Rightarrow \sigma = \frac{G_F^2 s}{\pi}$$

The cross section grows linearly with  $s$ . This is hardly surprising: the

coupling  $G_F$  has units of  $(\text{energy})^{-2}$  so on dimensional grounds the cross section must go like  $G_F^2$  times something with units of  $(\text{energy})^2$ .

Although seemingly innocuous, this sort of power-law growth of a cross-section is unacceptable. A good way to make this statement precise is to study scattering of states with definite total angular momenta. That is, in place of the scattering angle  $\theta$ , we'll specify the total angular momentum  $J$ . As discussed in appendix B the partial wave decomposition of a scattering amplitude is<sup>†</sup>

$$f(\theta) = \frac{1}{i\sqrt{s}} \sum_{J=0}^{\infty} (2J+1) P_J(\cos \theta) \langle f | S_J(E) | i \rangle. \quad (8.5)$$

Here we're working in the center of mass frame, with energy  $E$  and angular momentum  $J$ .  $P_J$  is a Legendre polynomial and  $\theta$  is the center of mass scattering angle.  $|i\rangle$  and  $|f\rangle$  are the initial and final states, normalized to  $\langle i | i \rangle = \langle f | f \rangle = 1$ .  $S_J(E)$  is the  $S$ -matrix in the sector with energy  $E$  and angular momentum  $J$ . This amplitude is related to the center-of-mass differential cross section by

$$\left( \frac{d\sigma}{d\Omega} \right)_{\text{cm}} = |f(\theta)|^2.$$

The partial wave decomposition of the cross section is then

$$\sigma = \int d\Omega |f(\theta)|^2 = \frac{4\pi}{s} \sum_{J=0}^{\infty} (2J+1) \left| \langle f | S_J(E) | i \rangle \right|^2$$

where we used  $\int d\Omega P_J(\cos \theta) P_{J'}(\cos \theta) = \frac{4\pi}{2J+1} \delta_{JJ'}$ . This expresses the total cross section as a sum over partial waves,  $\sigma = \sum_J \sigma_J$  where

$$\sigma_J = \frac{4\pi}{s} (2J+1) \left| \langle f | S_J(E) | i \rangle \right|^2.$$

The  $S$ -matrix is unitary, so  $|f\rangle$  and  $S_J(E)|i\rangle$  are both unit vectors, and their inner product must satisfy  $|\langle f | S_J(E) | i \rangle| \leq 1$ . This gives an upper bound on the partial wave cross sections, namely

$$\sigma_J \leq \frac{4\pi}{s} (2J+1).$$

This result is actually quite general: as discussed in appendix B, it holds for high-energy scattering of states with arbitrary helicities.

<sup>†</sup> This formula is valid for inelastic scattering at high energies, with initial particles that are either spinless or have identical helicities, and final particles that are either spinless or have identical helicities. The general decomposition is given in appendix B.

To apply this to inverse muon decay we first need the cross section for polarized scattering  $\nu_{\mu L} e_L^- \rightarrow \mu_L^- \nu_{e L}$ . That's easy, we just multiply our spin-averaged cross section by 2 to undo the average over electron spins. To find the partial wave decomposition note that the IMD cross section is independent of  $\theta$ , so only the  $J = 0$  partial wave contributes in (8.5) and unitarity requires

$$\sigma = \frac{2G_F^2 s}{\pi} \leq \frac{4\pi}{s}.$$

This bound is saturated when

$$\sqrt{s} = (2\pi^2/G_F^2)^{1/4} = 610 \text{ GeV}.$$

What should we make of this? In principle there are three options for restoring unitarity.

- (i) It could be that perturbation theory breaks down and strong-coupling effects become important at this energy scale, which would just mean our tree-level estimate for the cross section is invalid.
- (ii) It could be that additional terms in the Lagrangian (operators with dimension 8, 10, ...) are important at this energy scale and need to be taken into account.
- (iii) It could be there are new degrees of freedom that become important at this energy scale. With a bit of luck, the whole theory might remain weakly coupled even at high energies.

It's hard to say anything definite about the first two possibilities. Fortunately it's possibility #3 that turns out to be realized.

## References

4-Fermi theory is discussed in section 6.1 of Quigg. There's a brief treatment in Cheng & Li section 11.1. The partial wave decomposition of a helicity amplitude is given in appendix B. It's also mentioned by Quigg on p. 95 and by Cheng & Li on p. 343.

## Exercises

### 8.1 Inverse muon decay

Consider the following 4-Fermi coupling.

$$\mathcal{L} = -\frac{G_F}{\sqrt{|g_V|^2 + |g_A|^2}} \bar{\mu} \gamma^\lambda (g_V - g_A \gamma^5) \nu_\mu \bar{\nu}_e \gamma_\lambda (1 - \gamma^5) e + \text{h.c.}$$

Here  $g_V$  and  $g_A$  are two coupling constants (‘vector’ and ‘axial-vector’), which can be complex in general. The standard model values are  $g_V = g_A = 1$  in which case only left-handed particles (and their right-handed antiparticles) participate in weak interactions.

- (i) Compute  $\langle |\mathcal{M}|^2 \rangle$  for the ‘inverse muon decay’ reactions

$$\begin{aligned} \nu_{\mu L} e^- &\rightarrow \mu^- \nu_e \\ \nu_{\mu R} e^- &\rightarrow \mu^- \nu_e \end{aligned}$$

The  $\nu_\mu$  is polarized (either left-handed or right-handed), but you should sum over the spins of all the other particles and divide by 2 to average over the electron spin. The easiest way to compute a spin-polarized amplitude for a massless particle is probably to insert chiral projection operators  $\frac{1}{2}(1 \pm \gamma^5)$  in front of the  $\nu_\mu$  field, as in Peskin and Schroeder p. 142.

- (ii) Suppose the incoming  $\nu_\mu$  beam contains a fraction  $n_L$  of left-handed neutrinos and  $n_R$  of right-handed neutrinos ( $n_L + n_R = 1$ ). What is the center-of-mass differential cross section for  $\nu_\mu e^- \rightarrow \mu^- \nu_e$ ? You can neglect the electron mass but should keep track of the muon mass. Express your answer in terms of  $\theta$  and  $\lambda$  where  $\theta$  is the angle between the outgoing muon and the beam direction and

$$\lambda = \frac{2 \operatorname{Re}(g_V g_A^*)}{|g_V|^2 + |g_A|^2}.$$

See Fig. 3 in Mishra et. al., *Phys. Rev. Lett.* **63** (1989) 132.

### 8.2 Pion decay

- (i) Derive the Noether currents  $j_L^{\mu a}, j_R^{\mu a}$  associated with the  $SU(2)_L \times SU(2)_R$  symmetry

$$\delta\psi_L = -\frac{i}{2} \lambda_L^a \sigma^a \psi_L \quad \delta\psi_R = -\frac{i}{2} \lambda_R^a \sigma^a \psi_R$$

of the strong interactions with two flavors of massless quarks. We’ll mostly be interested in the vector and axial-vector linear combinations  $j_V^{\mu a} = j_L^{\mu a} + j_R^{\mu a}, j_A^{\mu a} = -j_L^{\mu a} + j_R^{\mu a}$ .

- (ii) Repeat part (i) for the  $SU(2)$  non-linear  $\sigma$ -model

$$\mathcal{L} = \frac{1}{4} f_\pi^2 \text{Tr} \left( \partial_\mu U^\dagger \partial^\mu U \right)$$

where the symmetry is

$$\delta U = -\frac{i}{2} \lambda_L^a \sigma^a U + U \frac{i}{2} \lambda_R^a \sigma^a.$$

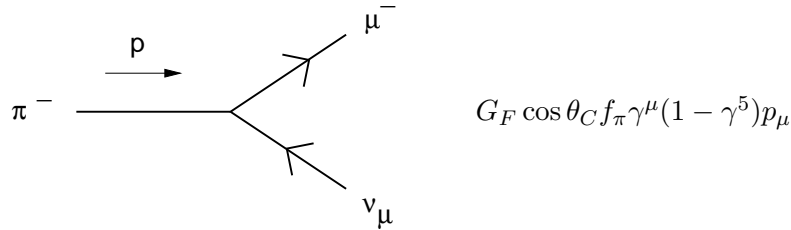
You only need to work out the symmetry currents to first order in the pion fields  $\pi^a$ , where  $U = e^{i\pi^a \sigma^a / f_\pi}$ .<sup>†</sup>

- (iii) The weak interaction responsible for the decay  $\pi^- \rightarrow \mu^- \bar{\nu}_\mu$  is

$$\mathcal{L}_{\text{weak}} = -\frac{1}{\sqrt{2}} G_F \cos \theta_C \bar{\mu} \gamma^\lambda (1 - \gamma^5) \nu_\mu \bar{u} \gamma_\lambda (1 - \gamma^5) d + h.c.$$

Here  $\theta_C \approx 13^\circ$  is the ‘Cabibbo angle.’ Suppose we can identify the symmetry currents worked out in parts (i) and (ii). Use this to rewrite  $\mathcal{L}_{\text{weak}}$  in terms of the fields  $\mu$ ,  $\nu_\mu$ ,  $\pi^a$ , again working to first order in the pion fields.

- (iv) If I did it right this leads to a vertex



Calculate the pion lifetime in terms of  $G_F$ ,  $\theta_C$ ,  $f_\pi$ ,  $m_\pi$ ,  $m_\mu$ . Given  $f_\pi = 93 \text{ MeV}$ , what's the pion lifetime? How did you do compared to the observed value  $2.6 \times 10^{-8} \text{ sec}$ ?

- (v) The decay  $\pi^- \rightarrow e^- \bar{\nu}_e$  only differs by replacing  $\mu \rightarrow e$ ,  $\nu_\mu \rightarrow \nu_e$ . Predict the branching ratio

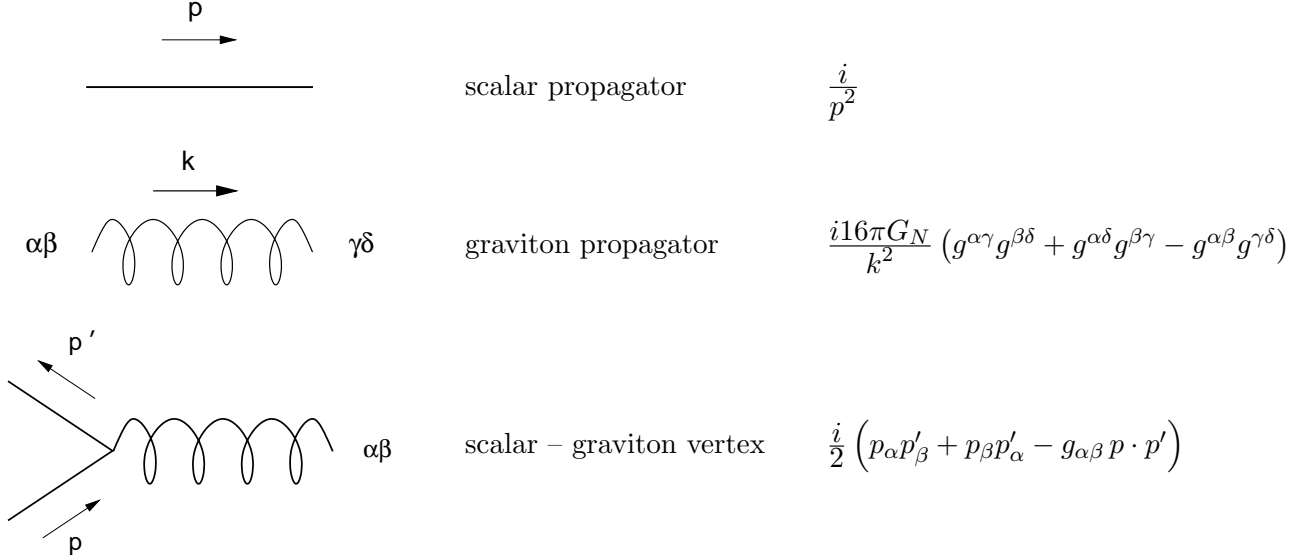
$$\frac{\Gamma(\pi^- \rightarrow e^- \bar{\nu}_e)}{\Gamma(\pi^- \rightarrow \mu^- \bar{\nu}_\mu)}$$

How well did you do?

<sup>†</sup> Given your expression for the currents in terms of the pion fields, the relation  $\langle \pi^a(p) | j_A^{\mu b}(x) | 0 \rangle = i f_\pi \delta^{ab} p^\mu e^{ip \cdot x}$  given in (5.6) follows.

### 8.3 Unitarity violation in quantum gravity

Consider two distinct types of massless scalar particles  $A$  and  $B$  which only interact gravitationally. The Feynman rules are



Here  $G_N = 6.7 \times 10^{-39} \text{ GeV}^{-2}$  is Newton's constant and  $g_{\alpha\beta} = \text{diag}(+---)$  is the Minkowski metric. The vertices and propagators are the same whether the scalar particle is of type  $A$  or type  $B$ .

- (i) Compute the tree-level amplitude and center-of-mass differential cross section for the process  $AA \rightarrow BB$ .
- (ii) The partial-wave expansion of the scattering amplitude is

$$f(\theta) = \frac{1}{i\sqrt{s}} \sum_{l=0}^{\infty} (2l+1) P_l(\cos \theta) S_l(E)$$

where  $P_l$  is a Legendre polynomial. This is related to the center-of-mass differential cross section by

$$\left( \frac{d\sigma}{d\Omega} \right)_{\text{cm}} = |f(\theta)|^2.$$

Compute the partial-wave S-matrix elements  $S_l(E)$ . For which values of  $l$  are they non-zero?

- (iii) At what center-of-mass energy is the unitarity bound  $|S_l(E)| \leq 1$  violated?

## 9

### Intermediate vector bosons

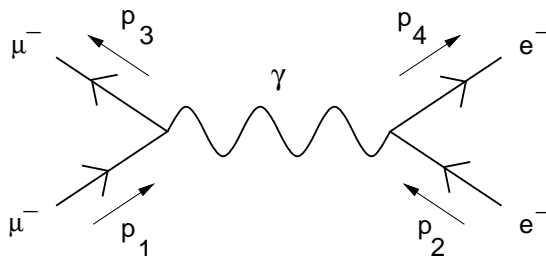
Physics 85200

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#### 9.1 Intermediate vector bosons

We'd like to regard the 4-Fermi theory of weak interactions as an effective low-energy approximation to some more fundamental theory in which – at an energy of order  $1/\sqrt{G_F}$  – new degrees of freedom become important and cure the problems of 4-Fermi theory.

A good example to keep in mind is the  $\phi^2\chi$  theory discussed in chapter 7, where exchange of a massive  $\chi$  particle between  $\phi$  quanta gave rise to an effective  $\phi^4$  interaction at low energies. For further inspiration recall the QED amplitude for  $\mu^-e^- \rightarrow \mu^-e^-$  elastic scattering.

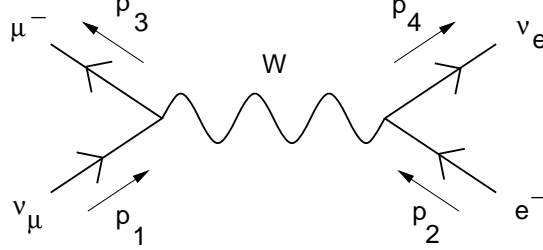


$$-i\mathcal{M} = \bar{u}(p_3)(-ieQ\gamma_\mu)u(p_1)\frac{-ig^{\mu\nu}}{(p_1 - p_3)^2}\bar{u}(p_4)(-ieQ\gamma_\nu)u(p_2)$$

The amplitude is built from two vector currents connected by a photon propagator. We saw this diagram “on its side,” when we calculated the QED cross section for  $e^+e^- \rightarrow \mu^+\mu^-$  and obtained the well-behaved result

$$\left(\frac{d\sigma}{d\Omega}\right)_{e^+e^- \rightarrow \mu^+\mu^-} = \frac{e^4}{64\pi^2 s} (1 + \cos^2 \theta) .$$

This suggests that to describe inverse muon decay  $\nu_\mu e^- \rightarrow \mu^- \nu_e$  we should pull apart the 4-Fermi vertex and write the IMD amplitude as



$$-i\mathcal{M} = \bar{u}(p_3) \left( -\frac{ig}{2\sqrt{2}} \gamma_\mu (1-\gamma^5) \right) u(p_1) D^{\mu\nu}(p_1-p_3) \bar{u}(p_4) \left( -\frac{ig}{2\sqrt{2}} \gamma_\nu (1-\gamma^5) \right) u(p_2) \quad (9.1)$$

In this expression  $g$  is the weak coupling constant; the factors of  $1/2\sqrt{2}$  are included to match the conventions of the standard model. The two “charged weak currents” are assumed to have a  $V-A$  form in which only left-handed chiral spinors enter. Finally  $D^{\mu\nu}(k)$  is the propagator for a new degree of freedom: an intermediate vector boson  $W^\pm$ .

To reproduce the successes of 4-Fermi theory the  $W^\pm$  must have some unusual properties.

- (i) It must be massive, with  $m_W = \mathcal{O}(1/\sqrt{G_F})$ , so that at energies  $\ll m_W$  we recover the point-like interaction of 4-Fermi theory.
- (ii) It must carry  $\pm 1$  unit of electric charge, so that electric charge is conserved at each vertex in the IMD diagram.
- (iii) It must have spin 1 so that it can couple to the Lorentz vector index on the  $V-A$  currents.

## 9.2 Massive vector fields

At this point we need to develop the field theory of a free (non-interacting) massive vector particle. This material can be found in Mandl & Shaw chapter 11.

Let's begin with a vector field  $W_\mu$  which we'll take to be complex since we want to describe charged particles. The field strength of  $W_\mu$  is defined in the usual way,  $G_{\mu\nu} = \partial_\mu W_\nu - \partial_\nu W_\mu$ . The Lagrangian is

$$\mathcal{L} = -\frac{1}{2} G_{\mu\nu}^* G^{\mu\nu} + m_W^2 W_\mu^* W^\mu$$



Aside from the mass term this looks a lot like a (complex version of) electromagnetism. The equations of motion from varying the action are

$$\partial_\mu G^{\mu\nu} + m_W^2 W^\nu = 0.$$

Acting on this with  $\partial_\nu$  implies  $\partial_\mu W^\mu = 0$ . Then  $\partial_\mu G^{\mu\nu} = \partial_\mu(\partial^\mu W^\nu - \partial^\nu W^\mu) = \square W^\nu$  and the equations of motion can be summarized as

$$\begin{aligned} (\square + m_W^2) W^\mu &= 0 && \text{a set of decoupled massive wave equations} \dots \\ \partial_\mu W^\mu &= 0 && \dots \text{obeying a "Lorentz gauge" condition} \end{aligned}$$

(When  $m_W \neq 0$  this theory does not have a gauge symmetry. The "Lorentz gauge" condition is an equation of motion, not a gauge choice.)

There are three independent polarization vectors that satisfy the Lorentz condition  $\epsilon \cdot k = 0$  with  $k^2 = m_W^2$ . For a W-boson moving in the  $+z$  direction with

$$k^\mu = (\omega, 0, 0, k) \quad \omega \equiv \sqrt{k^2 + m_W^2}$$

a convenient basis of polarization vectors is

$$\begin{aligned} \epsilon^\pm &= \frac{1}{\sqrt{2}}(0, 1, \pm i, 0) && \text{two transverse polarizations} \\ \epsilon^0 &= \frac{1}{m_W}(k, 0, 0, \omega) && \text{longitudinal polarization} \end{aligned}$$

The transverse polarizations have helicity  $\pm 1$ , while the longitudinal polarization has helicity 0. These obey the orthogonality / completeness relations

$$\sum_\mu \epsilon_\mu^{i*} \epsilon^{j\mu} = -\delta^{ij} \quad \sum_i \epsilon_\mu^i \epsilon_\nu^{i*} = -g_{\mu\nu} + \frac{k_\mu k_\nu}{m_W^2}$$

To get the W-boson propagator we first integrate by parts to rewrite the Lagrangian as

$$\mathcal{L} = W_\mu^* \mathcal{O}^{\mu\nu} W_\nu \quad \mathcal{O}^{\mu\nu} = (\square + m_W^2)g^{\mu\nu} - \partial^\mu \partial^\nu$$

Following a general rule the vector boson propagator is  $i$  times the inverse of the operator that appears in the quadratic part of the Lagrangian:  $D_{\mu\nu} = i(\mathcal{O}^{-1})_{\mu\nu}$ . To compute the inverse we go to momentum space,

$$\mathcal{O}^\mu{}_\nu(k) = (-k^2 + m_W^2)\delta^\mu{}_\nu + k^\mu k_\nu.$$

Note that, regarded as a  $4 \times 4$  matrix,  $\mathcal{O}$  has eigenvalue  $-k^2 + m_W^2$  when it acts on any vector orthogonal to  $k$ , and eigenvalue  $m_W^2$  when it acts on  $k$

itself. Then with the help of some projection operators

$$\begin{aligned}
(\mathcal{O}^{-1})^\mu{}_\nu &= \frac{1}{-k^2 + m_W^2} \left( \delta^\mu{}_\nu - \frac{k^\mu k_\nu}{k^2} \right) + \frac{1}{m_W^2} \frac{k^\mu k_\nu}{k^2} \\
&= \frac{1}{-k^2 + m_W^2} \left[ \delta^\mu{}_\nu - \frac{k^\mu k_\nu}{k^2} + \frac{(-k^2 + m_W^2)k^\mu k_\nu}{m_W^2 k^2} \right] \\
&= \frac{1}{-k^2 + m_W^2} \left[ \delta^\mu{}_\nu - \frac{k^\mu k_\nu}{m_W^2} \right]
\end{aligned}$$

and the propagator is

$$D_{\mu\nu}(k) = i\mathcal{O}_{\mu\nu}^{-1}(k) = \frac{-i(g_{\mu\nu} - \frac{k_\mu k_\nu}{m_W^2})}{k^2 - m_W^2}$$

### 9.3 Inverse muon decay revisited

Now that we know the  $W$  propagator, the amplitude for inverse muon decay is

$$\mathcal{M} = \frac{g^2}{8} \bar{u}(p_3) \gamma^\mu (1 - \gamma^5) u(p_1) \frac{-g_{\mu\nu} + \frac{k_\mu k_\nu}{m_W^2}}{k^2 - m_W^2} \bar{u}(p_4) \gamma^\nu (1 - \gamma^5) u(p_2).$$

Here  $k = p_1 - p_3$ . First let's consider the low energy behavior. At small  $k$  the factor in the middle from the  $W$  propagator reduces to  $g_{\mu\nu}/m_W^2$  and the amplitude becomes

$$\mathcal{M} = \frac{g^2}{8m_W^2} \bar{u}(p_3) \gamma^\mu (1 - \gamma^5) u(p_1) \bar{u}(p_4) \gamma_\mu (1 - \gamma^5) u(p_2) \quad (9.2)$$

This reproduces our old 4-Fermi amplitude (8.3) provided we identify

$$G_F = g^2/4\sqrt{2}m_W^2. \quad (9.3)$$

Now let's see what happens at high energies. We can regard the amplitude as a sum of two terms,  $\mathcal{M} = \mathcal{M}_1 + \mathcal{M}_2$ , where  $\mathcal{M}_1$  comes from the  $g_{\mu\nu}$  part of the  $W$  propagator and  $\mathcal{M}_2$  comes from the  $k_\mu k_\nu/m_W^2$  part of the  $W$  propagator. Let's look at  $\mathcal{M}_1$  first.

$$\begin{aligned}
\mathcal{M}_1 &= -\frac{g^2}{8(k^2 - m_W^2)} \bar{u}(p_3) \gamma^\mu (1 - \gamma^5) u(p_1) \bar{u}(p_4) \gamma_\mu (1 - \gamma^5) u(p_2) \\
&= \frac{-m_W^2}{k^2 - m_W^2} \cdot \frac{1}{\sqrt{2}} G_F \bar{u}(p_3) \gamma^\mu (1 - \gamma^5) u(p_1) \bar{u}(p_4) \gamma_\mu (1 - \gamma^5) u(p_2)
\end{aligned}$$

This is our old 4-Fermi amplitude (8.3) times a factor  $-m_W^2/(k^2 - m_W^2)$ .

The extra factor goes to one at small  $k$  and suppresses the amplitude at large  $k$ . So far, so good. However the other contribution to the amplitude is

$$\mathcal{M}_2 = \frac{g^2}{8m_W^2(k^2 - m_W^2)} \bar{u}(p_3) \not{k} (1 - \gamma^5) u(p_1) \bar{u}(p_4) \not{k} (1 - \gamma^5) u(p_2)$$

At first glance this doesn't look suppressed at large  $k$ , but here we get lucky: it's not only suppressed at large  $k$ , it's negligible compared to  $\mathcal{M}_1$ . To see this note that

$$\begin{aligned} \bar{u}(p_3) \not{k} (1 - \gamma^5) u(p_1) &= \bar{u}(p_3) (1 + \gamma^5) \not{p}_1 u(p_1) - \bar{u}(p_3) \not{p}_3 (1 - \gamma^5) u(p_1) \\ &= -m_\mu \bar{u}(p_3) (1 - \gamma^5) u(p_1) \end{aligned}$$

where in the second step we used the Dirac equation for the external line factors

$$\not{p}_1 u(p_1) = 0 \quad \bar{u}(p_3) \not{p}_3 = \bar{u}(p_3) m_\mu$$

(the neutrino is massless!). Likewise we have

$$\bar{u}(p_4) \not{k} (1 - \gamma^5) u(p_2) = -m_e \bar{u}(p_4) (1 + \gamma^5) u(p_2)$$

which means that

$$\mathcal{M}_2 = \frac{g^2}{8(k^2 - m_W^2)} \frac{m_e m_\mu}{m_W^2} \bar{u}(p_3) (1 - \gamma^5) u(p_1) \bar{u}(p_4) (1 + \gamma^5) u(p_2).$$

So  $\mathcal{M}_2$  is not only suppressed at large  $k$ , it's down by a factor  $m_e m_\mu / m_W^2$  compared to  $\mathcal{M}_1$ .

To summarize, up to corrections of order  $m_e m_\mu / m_W^2$ , the amplitude for inverse muon decay is  $(4 - \text{Fermi}) \times (-m_W^2 / (k^2 - m_W^2))$ . Neglecting the electron and muon masses, the cross section is

$$\frac{d\sigma}{d\Omega} = \frac{G_F^2 m_W^4 s}{4\pi^2 (k^2 - m_W^2)^2} = \frac{G_F^2 m_W^4 s}{4\pi^2 (s \sin^2(\theta/2) + m_W^2)^2}$$

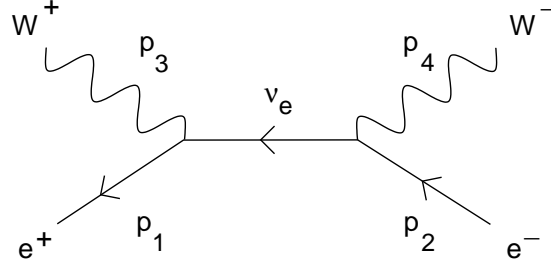
Fixed-angle scattering falls off like  $1/s$ . This is a huge improvement over the 4-Fermi cross section, and it's almost compatible with unitarity.<sup>†</sup>

## 9.4 Problems with intermediate vector bosons

So, have we succeeded in constructing a well-behaved theory of the weak interactions? Unfortunately the answer is no. Despite the nice features

<sup>†</sup> Scattering near  $\theta = 0$  isn't suppressed at large  $s$ , which in principle causes unitarity violations at incredibly large energies.

that intermediate vector bosons bring to the IMD amplitude, the theory has other problems. A classic example is  $e^+e^- \rightarrow W^+W^-$ , which in IVB theory is given by



$$-i\mathcal{M} = \bar{v}(p_1) \left( -\frac{ig}{2\sqrt{2}} \gamma_\mu (1 - \gamma^5) \right) \frac{i(\not{p}_1 - \not{p}_3)}{(p_1 - p_3)^2} \left( -\frac{ig}{2\sqrt{2}} \gamma_\nu (1 - \gamma^5) \right) u(p_2) \epsilon^\mu(p_3)^* \epsilon^\nu(p_4)^*$$

When you work out the amplitude in detail (Quigg p. 102) you find that the cross section grows linearly with  $s$ :

$$\frac{d\sigma}{d\Omega} = \frac{G_F^2 s \sin^2 \theta}{128\pi^2}.$$

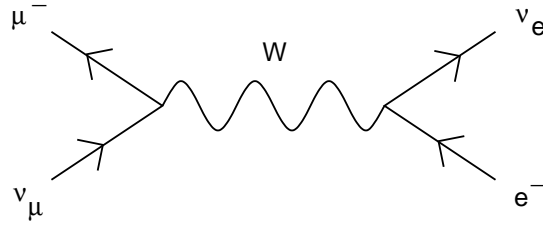
The cross section for producing transversely-polarized  $W$ 's is well-behaved; it's longitudinally-polarized  $W$ 's that cause trouble. Another way of seeing the difficulty with IVB theory is to note that the  $W$  propagator has bad high-energy behavior; it isn't suppressed at large  $k$  which leads to divergences in loops.

At this point the situation might seem a little hopeless; we've fixed inverse muon decay at the price of introducing problems somewhere else. Clearly we need a systematic procedure for constructing theories of spin-1 particles that are compatible with unitarity. Fortunately, such a procedure exists: theories based on local gauge symmetry turn out to have good high-energy behavior. More precisely, they're free from the sort of power-law growth in cross-sections that we encountered above.<sup>†</sup> We'll start constructing such theories in the next chapter.

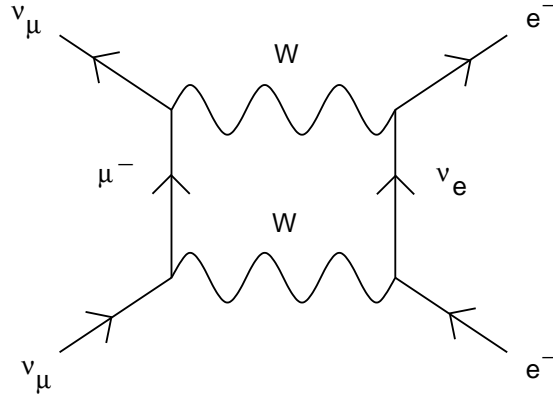
## 9.5 Neutral currents

There's one more ingredient I want to mention before we go on. We've spent a lot of time on inverse muon decay,  $\nu_\mu e^- \rightarrow \mu^- \nu_e$ .

<sup>†</sup> For more discussion of this point see Peskin & Schroeder, last paragraph of section 21.2.



But what about elastic scattering  $\nu_\mu e^- \rightarrow \nu_\mu e^-$ ? Based on what we've said so far we'd expect this to be a second-order process, a box diagram involving exchange of a pair of  $W$ 's.



If this is right then at energies much below  $m_W$  elastic scattering should be very suppressed (see below for an estimate). But in fact the low-energy cross sections for IMD and elastic scattering seem to have the same energy dependence and are roughly comparable in magnitude: measurements by the CHARM II collaboration give<sup>†</sup>

$$\frac{\sigma_{\text{elastic}}}{\sigma_{\text{IMD}}} \approx 0.09.$$

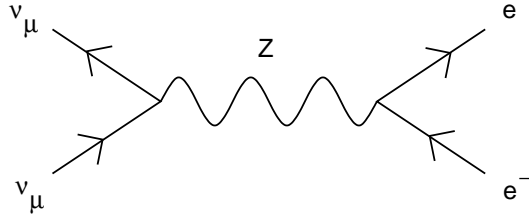
Similar behavior is seen in neutrino – nucleon scattering, where one finds<sup>‡</sup>

$$R_\nu = \frac{\sigma(\nu_\mu N \rightarrow \nu_\mu \text{anything})}{\sigma(\nu_\mu N \rightarrow \mu^- \text{anything})} \approx 0.31.$$

This means we need to postulate the existence of an electrically neutral IVB, the  $Z^0$ , which can mediate these sorts of processes at tree level.

<sup>†</sup> Phys. Lett. **B335** (1994) 246. The prefactor is  $(g_V^2 + g_A^2 + g_V g_A)/3$  where  $g_V = -0.035$  and  $g_A = -0.503$ .

<sup>‡</sup> CDHS collaboration, Z. Phys. **C45** (1990) 361.



Let's return to estimate the suppression factor associated with the box diagram for elastic scattering. First let's neglect the  $k_\mu k_\nu/m_W^2$  terms in the  $W$  propagator. This gives an amplitude that, for small external momenta, is schematically of the form

$$\mathcal{M} \sim g^4 \bar{u} \gamma^\mu (1 - \gamma^5) u \bar{u} \gamma_\mu (1 - \gamma^5) u \int \frac{d^4 k}{(2\pi)^4} \frac{1}{k^2 (k^2 - m_W^2)}.$$

The (Euclidean) loop integral gives (recall  $d^4 k_E = -i d^4 k$ ,  $k_E^2 = -k^2$ )

$$\int \frac{d^4 k_E}{(2\pi)^4} \frac{1}{k_E^2 (k_E^2 + m_W^2)^2} = \frac{1}{16\pi^2 m_W^2}.$$

So compared to the amplitude for inverse muon decay (9.2), which is schematically of the form  $g^2 \bar{u} \gamma^\mu (1 - \gamma^5) u \bar{u} \gamma_\mu (1 - \gamma^5) u / m_W^2$ , we'd expect the elastic scattering amplitude to be suppressed by a factor  $\sim g^2/16\pi^2$ . The cross section should then be suppressed by  $\sim (g^2/16\pi^2)^2 \sim 10^{-5}$ , where we've used the value of the weak coupling constant discussed in chapter 12. The  $k_\mu k_\nu/m_W^2$  terms that we neglected make a similar contribution, provided one cuts off the loop integral at  $k_E^2 \sim m_W^2$ .<sup>†</sup>

This calculation illustrates a general feature, that a factor  $g^2/16\pi^2$  is usually associated with each additional loop in a Feynman diagram. The factor of  $g^2$  can be understood from the topology of the diagram, while the numerical factor  $1/16\pi^2$  results from doing a typical loop integral.<sup>‡</sup>

## References

Intermediate vector bosons are discussed in section 6.2 of Quigg. They're also mentioned briefly in section 11.1 of Cheng & Li. For a nice field theory treatment of IVB's see chapter 11 of Mandl & Shaw, *Quantum field theory*.

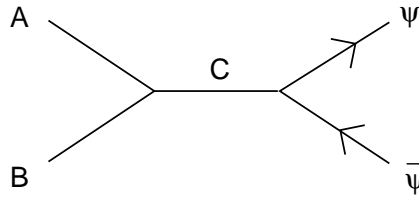
<sup>†</sup> A more careful procedure is to add an elementary 4-Fermi interaction to the theory and absorb these divergences by renormalizing the 4-Fermi coupling. This behavior is typical of non-renormalizable theories and illustrates the difficulties with loops in IVB theory.

<sup>‡</sup> For example, working in Euclidean space, another typical loop integral is  $\int \frac{d^4 p}{(2\pi)^4} \frac{1}{p^4} = \int_\mu^\Lambda \frac{2\pi^2 p^3 dp}{(2\pi)^4} \frac{1}{p^4} = \frac{1}{16\pi^2} \log \frac{\Lambda^2}{\mu^2}$ .

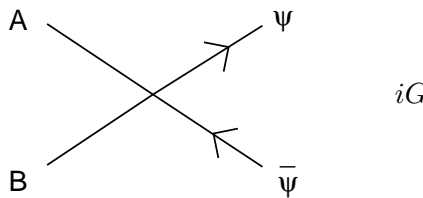
## Exercises

9.1 Unitarity and  $AB\psi$  theory

Recall the calculation of  $AB \rightarrow \psi\bar{\psi}$  scattering in the  $ABC\psi$  theory of problem A.1.



Suppose  $C$  has a very large mass. Then a low-energy observer would interpret this scattering in terms of an elementary quartic vertex with Feynman rule



This rule defines  $AB\psi$  theory – the “low energy effective theory” for the underlying  $ABC\psi$ .

- (i) Compute the amplitude for  $AB \rightarrow \psi\bar{\psi}$  scattering in  $AB\psi$  theory. There’s no need to average over spins at this stage.
- (ii) Match your result to the low-energy behavior of the same amplitude calculated in  $ABC\psi$  theory. Use this to fix the value of the coupling  $G$  in terms of  $g_1$ ,  $g_2$  and  $m_C$ .
- (iii) Find the differential cross-section for  $AB \rightarrow \psi_R\bar{\psi}_R$  in the low energy effective theory, where both outgoing particles are right-handed (positive helicity). How does the cross section behave at high energies? Which partial waves contribute? At what energy is unitarity violated?
- (iv) In the underlying  $ABC\psi$  theory, how does the same cross section behave at high energies? Is it compatible with unitarity?

# 10

## QED and QCD

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### 10.1 Gauge-invariant Lagrangians

After all these preliminaries we're finally ready to write down a Lagrangian which describes the electromagnetic and strong interactions of quarks. For the electromagnetic interaction it's easy: we introduce a collection of Dirac spinor fields

$$q_i \quad i = u, d, s \quad \text{for three flavors of quarks}$$

and couple them to electromagnetism in the standard way, via a Lagrangian

$$\mathcal{L}_{QED} = \sum_i \bar{q}_i \left[ i\gamma^\mu (\partial_\mu + ieQ_i A_\mu) - m_i \right] q_i - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \quad (10.1)$$

Here  $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$  is the Maxwell field strength and the various quarks have charges

$$Q_u = \frac{2}{3} \quad Q_d = Q_s = -\frac{1}{3}$$

in units of  $e = \sqrt{4\pi\alpha}$ .

There's a formal way of motivating this Lagrangian, which has the advantage of directly generalizing to the strong interactions. Start with three free quarks, described by

$$\begin{aligned} \mathcal{L}_{\text{free}} &= \mathcal{L}_{\text{kinetic}} + \mathcal{L}_{\text{mass}} \\ \mathcal{L}_{\text{kinetic}} &= \bar{Q} i\gamma^\mu \partial_\mu Q \quad \mathcal{L}_{\text{mass}} = -\bar{Q} M Q \\ Q &= \begin{pmatrix} u \\ d \\ s \end{pmatrix} \quad M = \begin{pmatrix} m_u & 0 & 0 \\ 0 & m_d & 0 \\ 0 & 0 & m_s \end{pmatrix} \end{aligned}$$



$\mathcal{L}_{\text{kinetic}}$  has a  $U(3)$  flavor symmetry  $Q \rightarrow UQ$  which is generically broken to  $U(1)^3$  by  $\mathcal{L}_{\text{mass}}$ . Let's focus on the particular transformation  $Q \rightarrow e^{-ie\alpha T}Q$  where  $\alpha$  is an angle that parametrizes the transformation and

$$T = \begin{pmatrix} 2/3 & 0 & 0 \\ 0 & -1/3 & 0 \\ 0 & 0 & -1/3 \end{pmatrix}$$

$T$  is a Hermitian matrix. It's one of the generators of the unbroken  $U(1)^3 \subset U(3)$  flavor symmetry.

Suppose we want to promote this global symmetry to a *local* invariance,  $\alpha \mapsto \alpha(x)$ . We can do this by “gauging” the symmetry, namely replacing the ordinary derivative  $\partial_\mu$  with a covariant derivative  $\mathcal{D}_\mu = \partial_\mu + ieA_\mu T$ . This replacement turns the free Lagrangian into

$$\mathcal{L}_{\text{gauged}} = \bar{Q} (i\gamma^\mu \mathcal{D}_\mu - M) Q.$$

This interacting Lagrangian has a local gauge invariance

$$Q(x) \rightarrow e^{-ie\alpha(x)T}Q(x) \quad A_\mu \rightarrow A_\mu + \partial_\mu \alpha.$$

To see this it's useful to note that  $\mathcal{D}_\mu Q$  transforms covariantly under gauge transformations (meaning in the same way as  $Q$  itself): that is  $\mathcal{D}_\mu Q \rightarrow e^{-ie\alpha(x)T}\mathcal{D}_\mu Q$ . Although this theory is perfectly gauge invariant, it lacks kinetic terms for the gauge fields. We can remedy this by adding the (gauge-invariant) Maxwell Lagrangian.

$$\mathcal{L}_{\text{Maxwell}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \quad (10.2)$$

This takes us back to the QED Lagrangian (10.1). One says that we have constructed this theory by “gauging a  $U(1)$  subgroup of the global symmetry group.”

How should we describe strong interactions of quarks? Inspired by electromagnetism, let's identify a global symmetry and gauge it. What global symmetry should we use? Recall that quarks come in three colors, so that each quark flavor is really a collection of three Dirac spinors.

$$q_i = \begin{pmatrix} q_{i,\text{red}} \\ q_{i,\text{green}} \\ q_{i,\text{blue}} \end{pmatrix}$$

Focusing for the moment on a single quark flavor, this means the free quark Lagrangian has an  $SU(3)_{\text{color}}$  symmetry. That is,

$$\mathcal{L}_{\text{free}} = \bar{q} (i\gamma^\mu \partial_\mu - m) q$$

is invariant under  $q \rightarrow Uq$  where  $U \in SU(3)$ . This symmetry is the basis for our theory of strong interactions (QCD, for “quantum chromodynamics”).

Let’s gauge  $SU(3)_{\text{color}}$ , following the procedure we used for electromagnetism. We’re after a local color symmetry

$$q(x) \rightarrow e^{-ig\alpha^a(x)T^a} q(x)$$

where we’ve introduced a coupling constant  $g$  and a set of eight  $3 \times 3$  traceless Hermitian matrices  $T^a$  – the generators of  $SU(3)_{\text{color}}$ . A gauge-invariant Lagrangian is

$$\mathcal{L}_{\text{gauged}} = \bar{q} (i\gamma^\mu \mathcal{D}_\mu - m) q$$

where the covariant derivative

$$\mathcal{D}_\mu = \partial_\mu + igB_\mu^a T^a$$

involves a collection of eight color gauge fields  $B_\mu^a$ . This Lagrangian is invariant under

$$q(x) \rightarrow U(x)q(x) \quad U(x) = e^{-ig\alpha^a(x)T^a}$$

provided the gauge fields transform according to

$$B_\mu(x) \rightarrow U(x)B_\mu(x)U^\dagger(x) + \frac{i}{g}(\partial_\mu U)U^\dagger.$$

Here  $B_\mu(x) = B_\mu^a(x)T^a$  is a traceless Hermitian matrix-valued field. To verify the invariance one should first show that under a gauge transformation  $\mathcal{D}_\mu q$  transforms covariantly,  $\mathcal{D}_\mu q \rightarrow U\mathcal{D}_\mu q$ .

To get a complete theory we need to add some gauge-invariant kinetic terms for the fields  $B_\mu$ . It’s not so obvious how to do this. The correct Lagrangian turns out to be

$$\mathcal{L}_{\text{Yang-Mills}} = -\frac{1}{2}\text{Tr}(G_{\mu\nu}G^{\mu\nu})$$

where the field strength associated with  $B_\mu$  is

$$G_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu + ig[B_\mu, B_\nu]$$

(the last term is a matrix commutator). This generalizes the Maxwell Lagrangian (10.2) to a non-abelian gauge group. Under a gauge transformation you can check that  $G_{\mu\nu}$  transforms covariantly in the adjoint representation:  $G_{\mu\nu} \rightarrow UG_{\mu\nu}U^\dagger$ . This transformation might seem surprising – in electrodynamics we’re used to the field strength being gauge invariant – but combined with the cyclic property of the trace it suffices to make the Yang-Mills Lagrangian gauge invariant.

Going back to three flavors, the strong and electromagnetic interactions of the  $u, d, s$  quarks are described by the following  $SU(3) \times U(1)$  gauge theory.

$$\mathcal{L} = \bar{Q} \left[ i\gamma^\mu (\partial_\mu + ieA_\mu T + igB_\mu^a T^a) - M \right] Q - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} \text{Tr} (G_{\mu\nu} G^{\mu\nu})$$

Here the  $U(1)$  generator of electromagnetism is

$$T = \begin{pmatrix} 2/3 & 0 & 0 \\ 0 & -1/3 & 0 \\ 0 & 0 & -1/3 \end{pmatrix}_{\text{flavor}} \otimes \mathbf{1}_{\text{color}}$$

while the  $SU(3)$  color generators are really

$$T^a = \mathbf{1}_{\text{flavor}} \otimes T_{\text{color}}^a$$

and the mass matrix is

$$M = \begin{pmatrix} m_u & 0 & 0 \\ 0 & m_d & 0 \\ 0 & 0 & m_s \end{pmatrix}_{\text{flavor}} \otimes \mathbf{1}_{\text{color}}.$$

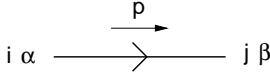
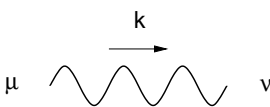
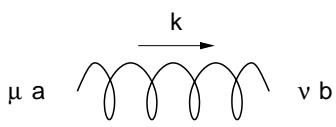
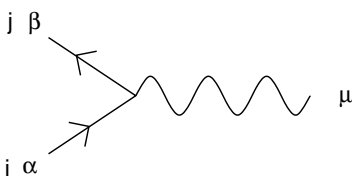
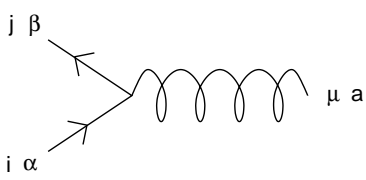
As we discussed in chapter 6, the free quark kinetic terms actually have an  $SU(3)_L \times SU(3)_R$  global flavor symmetry that acts on the chiral components of  $Q$ .

$$Q_L \rightarrow (L_{\text{flavor}} \otimes \mathbf{1}_{\text{color}}) Q_L \quad Q_R \rightarrow (R_{\text{flavor}} \otimes \mathbf{1}_{\text{color}}) Q_R \quad L, R \in SU(3)$$

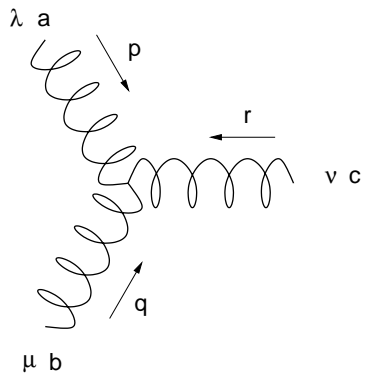
A very nice observation: if we neglect the quark masses and electromagnetic couplings, and take the gauge fields to be invariant, then the entire Lagrangian is invariant under this symmetry. This is just what we needed for our ideas about spontaneous chiral symmetry breaking by the strong interactions to make sense!

The Feynman rules are straightforward, at least at tree level.<sup>†</sup> One conventionally normalizes  $\text{Tr}(T^a T^b) = \frac{1}{2} \delta^{ab}$  and sets  $T^a = \frac{1}{2} \lambda^a$ ; the Gell-Mann matrices  $\lambda^a$  are the  $SU(3)$  analogs of the Pauli matrices. The Feynman rules are

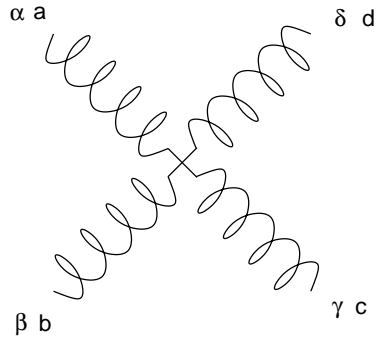
<sup>†</sup> Additional rules are needed to handle gluon loop diagrams.

	quark propagator	$\frac{i(\not{p} + m)}{p^2 - m^2} \delta_{ij} \delta_{\alpha\beta}$
	photon propagator	$\frac{-ig_{\mu\nu}}{k^2}$
	gluon propagator	$\frac{-ig_{\mu\nu}}{k^2} \delta_{ab}$
	quark - photon vertex	$-ieT_{ij}\delta_{\alpha\beta}\gamma^\mu$
	quark - gluon vertex	$-ig\delta_{ij}T_{\alpha\beta}^a\gamma^\mu$

Here  $i, j = u, d, s$  are quark flavor labels,  $\alpha, \beta = r, g, b$  are quark color labels,  $a = 1, \dots, 8$  is a gluon color label, and  $\mu$  is a Lorentz vector index denoting the photon or gluon polarization. Additional gluon 3-point and 4-point couplings arise from the cubic and quartic terms in the Yang-Mills action.



$$-gf^{abc} \left[ g_{\lambda\mu}(p - q)_\nu + g_{\mu\nu}(q - r)_\lambda + g_{\nu\lambda}(r - p)_\mu \right]$$

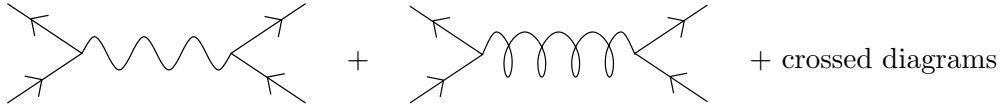


$$\begin{aligned}
 & -ig^2 \left[ f^{abe} f^{cde} (g_{\alpha\gamma} g_{\beta\delta} - g_{\alpha\delta} g_{\beta\gamma}) \right. \\
 & \quad + f^{ace} f^{bde} (g_{\alpha\beta} g_{\gamma\delta} - g_{\alpha\delta} g_{\gamma\beta}) \\
 & \quad \left. + f^{ade} f^{bce} (g_{\alpha\beta} g_{\delta\gamma} - g_{\alpha\gamma} g_{\delta\beta}) \right]
 \end{aligned}$$

The  $SU(3)$  structure constants are defined by  $[T^a, T^b] = if^{abc}T^c$ ; they're given explicitly in Quigg, p. 197. In the 4-gluon vertex  $\alpha, \beta, \gamma, \delta$  denote Lorentz vector indices – I hope that makes the structure of the vertex clearer.

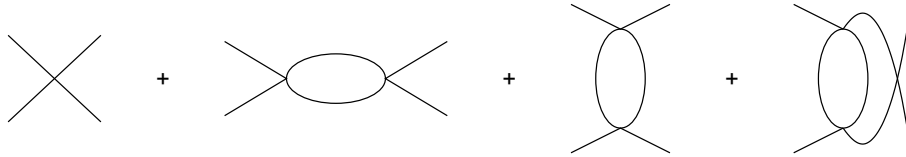
## 10.2 Running couplings

Given these Feynman rules it's straightforward (at least in principle) to do perturbative QCD calculations. For example, taking both photon and gluon exchange into account, at lowest order the quark – quark scattering amplitude is (time runs upwards)



You'll see these diagrams on the homework.

The most interesting thing one can compute in perturbation theory is the running coupling. It can be extracted from the behavior of scattering amplitudes. Recall that in  $\phi^4$  theory at one loop we studied the diagrams

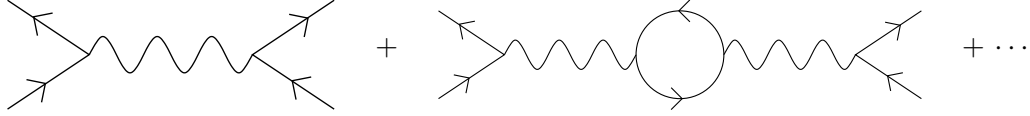


Evaluating these diagrams with a UV cutoff  $\Lambda$ , in section 7.2.1 we found the running coupling

$$\frac{1}{\lambda(\Lambda)} = \frac{1}{\lambda(\mu)} - \frac{3}{16\pi^2} \log \frac{\Lambda}{\mu}.$$

The coupling goes to zero in the infrared, and diverges in the UV at a scale  $\Lambda_{\max} = \mu e^{16\pi^2/3\lambda(\mu)}$ .

The analogous calculation in QED is to study  $e^- - e^-$  scattering at one loop, from the diagrams

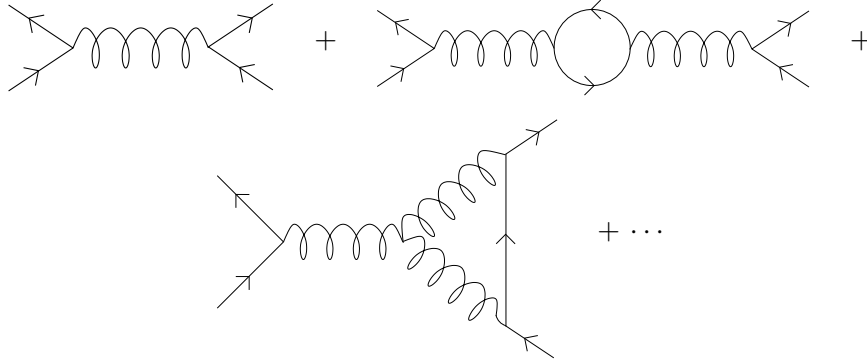


There are other one-loop diagrams that contribute to the scattering process, but the renormalization of electric charge arises solely from the “vacuum polarization” diagram drawn above.<sup>†</sup> As we saw in section 7.2.2 and problem 7.4, this leads to the running coupling

$$\frac{1}{e^2(\Lambda)} = \frac{1}{e^2(\mu)} - \frac{1}{6\pi^2} \log \frac{\Lambda}{\mu}.$$

Qualitatively, this is pretty similar to  $\phi^4$  theory: the coupling goes to zero in the infrared, and diverges at  $\Lambda_{\max} = \mu e^{6\pi^2/e^2(\mu)}$ .

In QCD quark – quark scattering arises from the diagrams



Several one-loop diagrams contribute to the running coupling; not all are shown. Summing them up leads to

$$\frac{1}{g^2(\Lambda)} = \frac{1}{g^2(\mu)} + \frac{11N_c - 2N_f}{24\pi^2} \log \frac{\Lambda}{\mu}.$$

Here  $N_c$  is the number of quark colors (three, in the real world) and  $N_f$  is the number of quark flavors (three if you count the light quarks, six if you include  $c, b, t$ ). A few comments:

<sup>†</sup> This is not to say there’s not a lot of interesting physics in the other diagrams. A detailed discussion can be found in Sakurai, *Advanced quantum mechanics*, section 4-7.

- In deriving the  $\beta$ -functions we neglected quark masses, so really  $N_f$  counts the number of quarks with  $m \ll \Lambda$ . As in problem 7.4, quarks with  $m \gg \Lambda$  don't contribute to the running.
- To recover the QED result from QCD one should set  $N_c = 0$  (to get rid of the extra non-abelian interactions),  $N_f = 1$  (since a single Dirac fermion runs around the electron bubble), and  $g^2 = 2e^2$  (compare the quark – gluon vertex for a  $U(1)$  gauge group, where our normalizations require  $T = 1/\sqrt{2}$ , to the quark – photon vertex).

The crucial point is that *the coefficient of the logarithm on the right hand side is positive*. This means the behavior of the QCD coupling is opposite to QED or  $\phi^4$  theory: the coupling goes to zero at short distances, and increases in the infrared. If you take the one-loop running seriously the renormalized coupling  $g^2(\mu)$  diverges when

$$\mu = \Lambda_{QCD} \equiv \Lambda e^{-24\pi^2/(11N_c - 2N_f)g^2(\Lambda)}.$$

This is known as the QCD scale. The notation  $\Lambda_{QCD}$  is standard, but as you can see it's not the same as the UV cutoff scale  $\Lambda$ .

The idea is that we can take the continuum limit by sending  $\Lambda \rightarrow \infty$  and  $g^2(\Lambda) \rightarrow 0$  while keeping  $\Lambda_{QCD}$  fixed. In this limit we should think of  $\Lambda_{QCD}$  as the unique (dimensionful!) quantity which characterizes the strong interactions. For example you can express the running coupling in terms of  $\Lambda_{QCD}$ .

$$\alpha_S(\mu) \equiv \frac{g^2(\mu)}{4\pi} = \frac{6\pi}{(11N_c - 2N_f) \log(\mu/\Lambda_{QCD})}$$

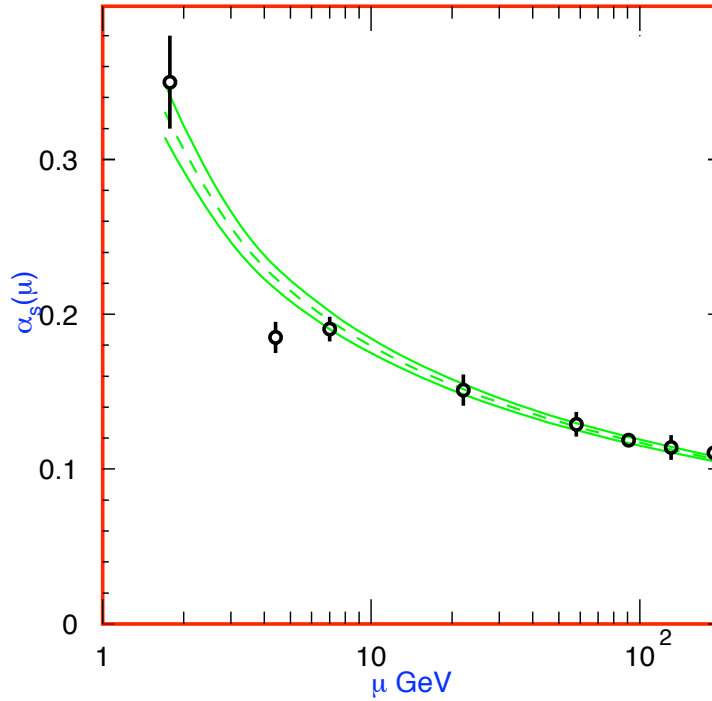
The particle data group (2002 version) gives the value  $\Lambda_{QCD} = 216^{+25}_{-24}$  MeV.

To summarize our basic picture of QCD:

- It's weakly coupled at short distances. In this regime perturbation theory can be trusted. For example gluon exchange gives rise to a short-distance Coulomb-like potential between quarks.
- It's strongly coupled at long distances. In this regime non-perturbative effects take over and give rise to phenomena such as quark confinement and spontaneous chiral symmetry breaking. In principle quantities that appear in the pion effective Lagrangian such as  $f$  and  $\mu$  can be calculated in terms of  $\Lambda_{QCD}$ .

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required, for example, to facilitate the extraction of CKM elements from measurements of charm and bottom decay rates. See Ref. 169 for a recent review.



**Figure 9.2:** Summary of the values of  $\alpha_s(\mu)$  at the values of  $\mu$  where they are measured. The lines show the central values and the  $\pm 1\sigma$  limits of our average. The figure clearly shows the decrease in  $\alpha_s(\mu)$  with increasing  $\mu$ . The data are, in increasing order of  $\mu$ ,  $\tau$  width,  $\Upsilon$  decays, deep inelastic scattering,  $e^+e^-$  event shapes at 22 GeV from the JADE data, shapes at TRISTAN at 58 GeV,  $Z$  width, and  $e^+e^-$  event shapes at 135 and 189 GeV.



## References

The basic material in this chapter is covered nicely in Quigg, chapter 4 and sections 8.1 – 8.3. Griffiths chapter 9 does some tree-level calculations in QCD. A more complete treatment can be found in Peskin & Schroeder: sections 15.1 and 15.2 work out the Yang-Mills action, section 16.1 gives the Feynman rules, and section 16.5 does the running coupling.

**GLUON LOOPS.** Additional Feynman rules are required to compute gluon loop diagrams. The additional rules ensure that unphysical gluon polarizations do not contribute in loops. The details are worked out in Peskin & Schroeder section 16.2.

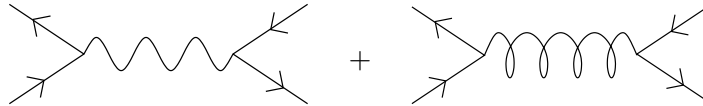
**PHOTON AND GLUON POLARIZATION SUMS.** The completeness relation we have been using to perform photon polarization sums,  $\sum_i \epsilon_\mu^i \epsilon_\nu^{i*} = -g_{\mu\nu}$ , implicitly requires a sum over four linearly independent polarization vectors (the two physical polarizations of a photon plus two unphysical polarizations). Such a sum can be used in QED: thanks to a cancellation discussed in Peskin & Schroeder p. 159, the unphysical polarizations do not contribute to scattering amplitudes. However the analogous cancellation does not always hold in QCD, so for gluons one should only sum over physical polarizations. The appropriate completeness relation is in Cheng & Li p. 271. The issue with gluon polarization sums is closely related to the additional rules for gluon loops, as nicely explained by Aitchison and Hey *Gauge theories in particle physics* (second edition, 1989) section 15.1.

**PARTONS.** The rules we have developed are adequate to describe the interactions of quarks and gluons. However to study scattering off a physical hadron one needs to work in terms of its constituent “partons.” The necessary machinery is developed in Peskin & Schroeder chapter 17.

## Exercises

### 10.1 Tree-level $q\bar{q}$ interaction potential

- (i) Compute the tree-level  $q\bar{q} \rightarrow q\bar{q}$  scattering amplitude arising from the one-photon and one-gluon exchange diagrams



(time runs upwards). Just write down the amplitude – you don’t

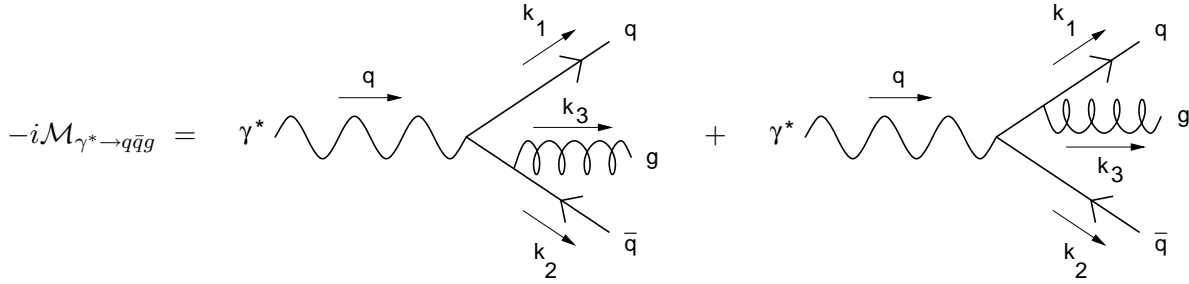
need to average over spins. We're assuming the quarks have distinct flavors so there's no diagram in which the  $q\bar{q}$  annihilate to an intermediate photon or gluon.

- (ii) The one-photon-exchange diagram generates the usual Coulomb potential  $V_{\text{QED}}(r) = \alpha Q_1 Q_2 / r$ . Comparing the normalization of the two diagrams, what is the analogous QCD potential  $V_{\text{QCD}}(r)$ ?
- (iii) Evaluate the QCD interaction potential when the  $q\bar{q}$  are in a color singlet state.

## 10.2 Three jet production

The process  $e^+e^- \rightarrow 3 \text{ jets}$  can be thought of as a two-step process,  $e^+e^- \rightarrow \gamma^*$  followed by  $\gamma^* \rightarrow q\bar{q}g$  where  $\gamma^*$  is an off-shell photon.

- (i) At leading order the diagrams for  $\gamma^* \rightarrow q\bar{q}g$  give



Compute  $\langle |\mathcal{M}_{\gamma^* \rightarrow q\bar{q}g}|^2 \rangle$ . You should average over the photon spin and sum over the spins, colors, and quark flavors in the final state. A few tips:

- You should allow the photon to be off-shell,  $q^2 \neq 0$ . However for simplicity you can take the other particles to be massless,  $k_i^2 = 0$ .
- You can sum over the photon and gluon spins using<sup>†</sup>

$$\sum_{\text{polarizations}} \epsilon_\mu^* \epsilon_\nu = -g_{\mu\nu}.$$

- To average over the photon spin you should divide your result by 3 for the three possible polarizations of a massive vector.
- You can sum over colors using  $\text{Tr } \lambda^a \lambda^b = 2\delta^{ab}$ .

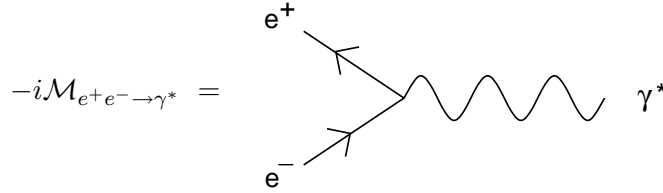
<sup>†</sup> You can't always perform gluon spin sums in such a simple way. See p. 113.

- You should express your answer in terms of the kinematic variables

$$x_i = \frac{2k_i \cdot q}{q^2}.$$

In the center of mass frame  $x_i$  is twice the energy fraction carried by particle  $i$ ,  $x_i = 2E_i/E_{\text{cm}}$ . Note that  $x_1 + x_2 + x_3 = 2$ .

- (ii) Compute the spin-averaged  $|\text{amplitude}|^2$  for  $e^+e^- \rightarrow \gamma^*$  from



- (iii) The spin-averaged  $|\text{amplitude}|^2$  for the whole process is

$$\langle |\mathcal{M}|^2 \rangle = \langle |\mathcal{M}_{e^+e^- \rightarrow \gamma^*}|^2 \rangle \cdot \frac{1}{q^4} \cdot \langle |\mathcal{M}_{\gamma^* \rightarrow q\bar{q}g}|^2 \rangle$$

where the  $1/q^4$  in the middle is from the intermediate photon propagator.<sup>†</sup> Plug this into the cross-section formula

$$\frac{d\sigma}{dx_1 dx_2} = \frac{1}{256\pi^3} \langle |\mathcal{M}|^2 \rangle$$

and find the differential cross-section for 3-jet events. You should reproduce Peskin & Schroeder (17.18).

<sup>†</sup> For a justification of this formula, including the factor of 3 for averaging over photon spins, see Peskin & Schroeder p. 261.

# 11

## Gauge symmetry breaking

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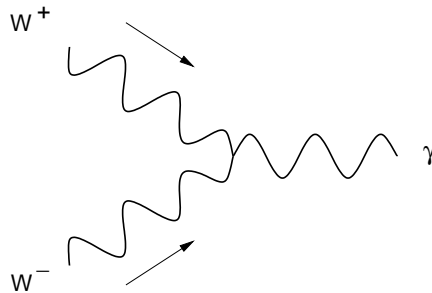
The only well-behaved theories of spin-1 particles are thought to be gauge theories. So we'd like to fit our IVB theory of weak interactions into the gauge theory framework. In trying to do this, there are a couple of obstacles.

- The Lagrangian for free  $W$ -bosons is

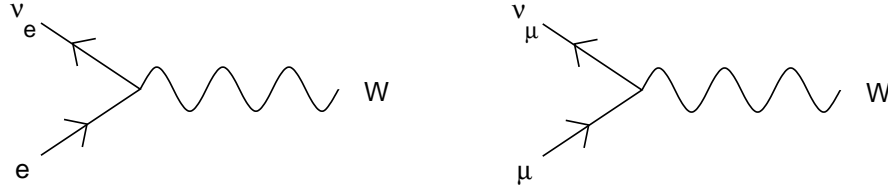
$$\mathcal{L} = -\frac{1}{2}G_{\mu\nu}^*G^{\mu\nu} + m_W^2 W_\mu^* W^\mu \quad G_{\mu\nu} = \partial_\mu W_\nu - \partial_\nu W_\mu.$$

However the mass term explicitly breaks the only real candidate for a gauge symmetry, namely invariance under  $W_\mu \rightarrow W_\mu + \partial_\mu \alpha$ .

- We know that  $W$ -bosons are charged, and should therefore couple to the photon.



This sounds like the sort of gauge boson self-interactions one has in non-abelian gauge theory. So it seems reasonable to look for an  $SU(2)$  (say) Yang-Mills theory of weak interactions. To match IVB theory we expect to find vertices



This suggests that we should group leptons and neutrinos into doublets under the gauge group.<sup>†</sup>

$$\begin{pmatrix} \nu_e \\ e \end{pmatrix} \quad \begin{pmatrix} \nu_\mu \\ \mu \end{pmatrix} \quad SU(2) \text{ doublets}$$

But it seems silly to group leptons and neutrinos in this way: they’re “obviously” not related by any type of symmetry, let alone gauge symmetry (they have different masses, charges, ...).

To write a gauge theory for the weak interactions we need a way of disguising the underlying gauge symmetry – the Lagrangian should be invariant, but the symmetry shouldn’t be manifest in the particle spectrum. We’re going to spontaneously break gauge invariance!

### 11.1 Abelian Higgs model

To illustrate the basic consequences of spontaneously breaking gauge invariance, let’s return to the model we used in chapter 5 to study spontaneous breaking of a continuous global symmetry.

$$\begin{aligned} \mathcal{L} &= \frac{1}{2} \partial_\mu \vec{\phi} \cdot \partial^\mu \vec{\phi} - \frac{1}{2} \mu^2 |\vec{\phi}|^2 - \frac{1}{4} \lambda |\vec{\phi}|^4 \\ &= \partial_\mu \phi^* \partial^\mu \phi - \mu^2 \phi^* \phi - \lambda (\phi^* \phi)^2 \end{aligned}$$

In the first line we’re working in terms of a real two component field  $\vec{\phi} = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$ , in the second line we introduced the complex combination  $\phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2)$ . Let’s gauge the global  $U(1)$  symmetry  $\phi \rightarrow e^{i\theta} \phi$ . Following the usual procedure we define a covariant derivative  $\mathcal{D}_\mu \phi = \partial_\mu \phi + ieA_\mu \phi$ . Replacing  $\partial_\mu \rightarrow \mathcal{D}_\mu$  and adding a Maxwell term gives a Lagrangian

$$\mathcal{L} = \mathcal{D}_\mu \phi^* \mathcal{D}^\mu \phi - \mu^2 \phi^* \phi - \lambda (\phi^* \phi)^2 - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

<sup>†</sup> This discussion is just to illustrate the idea; when we get to the standard model we’ll see that the actual gauge structure is somewhat different. Also the choice of  $SU(2)$  is just for simplicity – we could use a larger group and put more, possibly undiscovered, particles into the multiplets.

which is invariant under gauge transformations

$$\phi \rightarrow e^{-ie\alpha(x)}\phi \quad A_\mu \rightarrow A_\mu + \partial_\mu\alpha.$$

If  $\mu^2 > 0$  we have the usual situation: a massless photon coupled to a complex scalar field with quartic self-interactions. What happens for  $\mu^2 < 0$ ? In this case the potential has a circle of degenerate minima, located at

$$\phi^*\phi = -\mu^2/2\lambda, \quad \text{phase of } \phi \text{ arbitrary.}$$

Clearly something interesting is going to happen, because these vacua aren't really distinct: they're related by gauge transformations!

To see what's going on let's take a pedestrian approach, and expand about one of the minima. Without loss of generality we choose the minimum where  $\phi$  is real and positive, and set  $\phi = \frac{1}{\sqrt{2}}(\phi_0 + \rho)e^{i\theta}$ . Here  $\rho$  and  $\theta$  are real scalar fields and  $\phi_0 = \sqrt{-\mu^2/\lambda}$ . Then

$$\begin{aligned} \partial_\mu\phi &= \frac{1}{\sqrt{2}}\partial_\mu\rho e^{i\theta} + \frac{1}{\sqrt{2}}(\phi_0 + \rho)e^{i\theta}i\partial_\mu\theta \\ ieA_\mu\phi &= \frac{1}{\sqrt{2}}(\phi_0 + \rho)e^{i\theta}ieA_\mu \\ \hline \mathcal{D}_\mu\phi &= \frac{1}{\sqrt{2}}e^{i\theta}\left(\partial_\mu\rho + i(\phi_0 + \rho)(\partial_\mu\theta + eA_\mu)\right) \end{aligned}$$

and the Lagrangian becomes

$$\begin{aligned} \mathcal{L} &= \frac{1}{2}\partial_\mu\rho\partial^\mu\rho + \frac{1}{2}(\phi_0 + \rho)^2(\partial_\mu\theta + eA_\mu)(\partial^\mu\theta + eA^\mu) \\ &\quad - \frac{1}{2}\mu^2(\phi_0 + \rho)^2 - \frac{1}{4}\lambda(\phi_0 + \rho)^4 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \end{aligned}$$

This looks awfully complicated. To get a handle on what's going on let's expand to quadratic order in the fields, since we can identify the spectrum of particle masses by studying small oscillations about the minimum.

$$\begin{aligned} \mathcal{L} &= \frac{1}{2}\partial_\mu\rho\partial^\mu\rho + \frac{1}{2}\phi_0^2(\partial_\mu\theta + eA_\mu)(\partial^\mu\theta + eA^\mu) - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \\ &\quad + \text{const.} + \mu^2\rho^2 + \text{cubic, quartic interaction terms} \quad (11.1) \end{aligned}$$

The field  $\rho$  has a familiar mass,  $m_\rho^2 = -2\mu^2$ . The field  $\theta$  looks massless. That's no surprise, since we'd expect  $\theta$  to be the Goldstone boson associated with spontaneously breaking the  $U(1)$  symmetry. Expanding the terms in parenthesis it looks like there's a mass term for  $A_\mu$ , but it also looks like there are  $A_\mu\partial^\mu\theta$  cross-terms which give rise to mixing between  $A_\mu$  and  $\theta$ . In terms of diagrams there's a vertex



To understand what's going on, recall the gauge symmetry

$$\phi \rightarrow e^{-ie\alpha} \phi \quad A_\mu \rightarrow A_\mu + \partial_\mu \alpha.$$

In terms of  $\rho$  and  $\theta$  this becomes

$$\rho \text{ invariant} \quad \theta \rightarrow \theta - e\alpha \quad A_\mu \rightarrow A_\mu + \partial_\mu \alpha.$$

Let's choose  $\alpha(x) = \frac{1}{e}\theta(x)$ . In this so-called unitary gauge we have  $\theta = 0$ . The Lagrangian in unitary gauge is just given by setting  $\theta = 0$  in (11.1). Dropping a constant

$$\begin{aligned} \mathcal{L}_{\text{unitary}} = & \frac{1}{2} \partial_\mu \rho \partial^\mu \rho + \mu^2 \rho^2 - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} e^2 \phi_0^2 A_\mu A^\mu \\ & + \text{cubic, quartic interaction terms.} \end{aligned} \quad (11.2)$$

We ended up with a massive scalar field  $\rho$  coupled to a massive vector field  $A_\mu$ , and the would-be Goldstone boson  $\theta$  has disappeared! This is known as the Higgs mechanism. We can read off the masses

$$m_\rho^2 = -2\mu^2 \quad m_A^2 = e^2 \phi_0^2.$$

It's interesting to count the degrees of freedom in the two phases,

$$\begin{aligned} \mu^2 > 0: & \quad \text{two real scalars} \\ & \quad \text{massless photon (two polarizations)} \\ \mu^2 < 0: & \quad \text{one real scalar} \\ & \quad \text{massive photon (three polarizations)} \end{aligned}$$

In both cases there are a total of four degrees of freedom. A few comments:

- Due to the photon mass term, the Lagrangian (11.2) is not manifestly gauge invariant. But that's perfectly okay because  $\mathcal{L}_{\text{unitary}}$  is written in a particular gauge.
- Spontaneously broken gauge theories are renormalizable. This is hard to see in unitary gauge. It can be shown by working in a different class of gauges known as  $R_\xi$  gauges. See Peskin & Schroeder section 21.1.
- A related claim is that spontaneously broken gauge theories have well-behaved scattering amplitudes. In the abelian Higgs model a scattering process like  $\phi\phi \rightarrow AA$  should be compatible with unitarity. This will be discussed in the context of the standard model in section 14.1.

- In a very precise way one can identify the extra longitudinal polarization of the vector boson in the Higgs phase with the “eaten” Goldstone boson. See Peskin & Schroeder section 21.2.
- We’ve spontaneously broken an abelian gauge symmetry. The Higgs mechanism has a straightforward generalization to Yang-Mills theory, which we’ll see when we construct the standard model.

## References

The abelian Higgs model is discussed in Quigg section 5.3.

## Exercises

### 11.1 Superconductivity

Consider the abelian Higgs model at low energies, where we can ignore radial fluctuations in the Higgs field. In unitary gauge we set  $\phi = \phi_0/\sqrt{2}$  and the Lagrangian reduces to

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}e^2\phi_0^2 A_\mu A^\mu.$$

This is the free massive vector field discussed in section 9.2. It turns out to describe superconductivity.

- Compare the vector field equations of motion to Maxwell’s equations  $\partial_\mu F^{\mu\nu} = j^\nu$  and express the current  $j^\mu$  in terms of  $A^\mu$ . This is known as the London equation.
- Consider the following ansatz for a static solution to the equations of motion.

$$A^\mu = (V, \mathbf{A}) \quad V(t, \mathbf{x}) = 0 \quad \mathbf{A}(t, \mathbf{x}) = \mathbf{a}e^{-\mathbf{k}\cdot\mathbf{x}}$$

Here  $\mathbf{a}$  and  $\mathbf{k}$  are constant vectors. Show that this ansatz satisfies the equations of motion provided  $|\mathbf{k}|^2 = e^2\phi_0^2$  and  $\mathbf{a} \cdot \mathbf{k} = 0$ .

- Compute the electric and magnetic fields, and the current and charge densities, associated with this solution. (Recall  $\mathbf{E} = -\nabla V - \partial_0 \mathbf{A}$ ,  $\mathbf{B} = \nabla \times \mathbf{A}$ ,  $j^\mu = (\rho, \mathbf{j})$ .)

Comments: this exercise shows that spontaneously breaking an abelian gauge symmetry gives rise to superconductivity. Your solution illustrates the Meissner effect, that magnetic fields decay exponentially in a superconductor. The current also decays exponentially, showing



that currents in a superconductor are carried near the surface. Finally the resistance vanishes since we have a current with no electric field! (Recall Ohm's law  $\mathbf{J} = \sigma \mathbf{E}$  where  $\sigma$  is the conductivity.)

# 12

## The standard model

Physics 85200

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So far we've been taking a quasi-historical approach to the subject, constructing theories from the bottom up. Now we're going to switch to a top-down approach and derive the standard model from a set of postulates. We'll first discuss the electroweak interactions of a single generation of leptons, then treat the electroweak interactions of a single generation of quarks. Finally we'll put it all together in a 3-generation standard model.

### 12.1 Electroweak interactions of leptons

#### 12.1.1 The Lagrangian

The first order of business is to postulate a gauge group. To accommodate  $W^+, W^-, Z, \gamma$  we need a group with four generators. We'll take the gauge group to be

$$SU(2)_L \times U(1)_Y.$$

$SU(2)_L$  is only going to couple to left-handed spinors (hence the subscript  $L$ ), while  $U(1)_Y$  is a “hypercharge”  $U(1)$  gauge symmetry that should *not* be confused with the gauge group of electromagnetism. We'll see how electromagnetism emerges later on.

Next we need to postulate the matter content. At this point we'll focus on a single generation of leptons (the electron and the electron neutrino). We'll treat the left- and right-handed parts of the fields separately, and assign them the  $SU(2)_L \times U(1)_Y$  quantum numbers

$$\begin{aligned} L &= \begin{pmatrix} \nu \\ e \end{pmatrix}_L & SU(2)_L \text{ doublet with hypercharge } Y = -1 \\ R &= e_R & SU(2)_L \text{ singlet with hypercharge } Y = -2 \end{aligned}$$

Just to clarify the notation, this means the  $SU(2)_L$  generators  $T_L^a$  are

$$T_L^a = \begin{cases} \sigma^a/2 & \text{when acting on } L \\ 0 & \text{when acting on } R \end{cases}$$

while the hypercharge generator acts according to

$$YL = -L \quad YR = -2R.$$

Furthermore  $e_R$  is a right-handed Dirac spinor (that is, a Dirac spinor that is only non-zero in its bottom two components), while  $\nu_L$  and  $e_L$  are left-handed Dirac spinors. Note that the left- and right-handed spinors are assigned different  $U(1)_Y$  as well as  $SU(2)_L$  quantum numbers. Also note that we haven't introduced a right-handed neutrino  $\nu_R$ .

We need a mechanism for spontaneously breaking  $SU(2)_L \times U(1)_Y$  down to the  $U(1)$  gauge group of electromagnetism. The minimal way to accomplish this is to introduce a Higgs doublet

$$\phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} \quad SU(2)_L \text{ doublet with hypercharge } Y = +1.$$

Here  $\phi^+$  and  $\phi^0$  are complex scalar fields; as we'll see the superscripts indicate their electric charges. The standard model is defined as the most general renormalizable theory with these gauge symmetries and this matter content.<sup>†</sup>

It's straightforward to write down the standard model Lagrangian; it's the most general Lagrangian with operators up to dimension four. It's a sum of four terms,

$$\mathcal{L}_{SM} = \mathcal{L}_{\text{Dirac}} + \mathcal{L}_{\text{Yang-Mills}} + \mathcal{L}_{\text{Higgs}} + \mathcal{L}_{\text{Yukawa}}.$$

$\mathcal{L}_{\text{Dirac}}$  contains gauge-invariant kinetic terms for the fermions,

$$\begin{aligned} \mathcal{L}_{\text{Dirac}} &= \bar{L} i \gamma^\mu \mathcal{D}_\mu L + \bar{R} i \gamma^\mu \mathcal{D}_\mu R \\ &= \bar{L} i \gamma^\mu \left( \partial_\mu + \frac{ig}{2} W_\mu^a \sigma^a + \frac{ig'}{2} B_\mu Y \right) L + \bar{R} i \gamma^\mu \left( \partial_\mu + \frac{ig'}{2} B_\mu Y \right) R. \end{aligned}$$

In the second line we've written out the covariant derivatives explicitly.  $W_\mu^a$  are the  $SU(2)_L$  gauge fields, with generators  $T_L^a$  and coupling constant  $g$ . Also  $B_\mu$  is the hypercharge gauge field, with generator  $Y$  and coupling constant  $g'/2$ .<sup>‡</sup>

<sup>†</sup> There's no real reason to insist on renormalizability, and we will explore what happens when you add higher-dimension operators to the standard model Lagrangian. Also it might be worth pointing out something we *didn't* postulate, namely lepton number conservation. As we'll see, lepton number is conserved due to an "accidental symmetry."

<sup>‡</sup> The peculiar normalization of  $g'$  is chosen for later convenience.

$\mathcal{L}_{\text{Yang-Mills}}$  contains the gauge kinetic terms,

$$\mathcal{L}_{\text{Yang-Mills}} = -\frac{1}{2}\text{Tr}(W_{\mu\nu}W^{\mu\nu}) - \frac{1}{4}B_{\mu\nu}B^{\mu\nu}$$

where

$$W_{\mu\nu} = \frac{1}{2}W_{\mu\nu}^a\sigma^a = \partial_\mu W_\nu - \partial_\nu W_\mu + ig[W_\mu, W_\nu]$$

is the  $SU(2)_L$  field strength, built from the gauge fields  $W_\mu = \frac{1}{2}W_\mu^a\sigma^a$ , and

$$B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$$

is the  $U(1)_Y$  field strength.

$\mathcal{L}_{\text{Higgs}}$  includes gauge-covariant kinetic terms plus a potential for  $\phi$ .

$$\mathcal{L}_{\text{Higgs}} = \mathcal{D}_\mu\phi^\dagger\mathcal{D}^\mu\phi + \mu^2\phi^\dagger\phi - \lambda(\phi^\dagger\phi)^2$$

Here  $\mathcal{D}_\mu\phi = \partial_\mu\phi + \frac{ig}{2}W_\mu^a\sigma^a\phi + \frac{ig'}{2}B_\mu Y\phi$ . We'll assume that  $\mu^2 > 0$  so that  $\phi$  acquires a vev.

There's one more term we can write down. Note that  $\phi^\dagger L$  is an  $SU(2)_L$  singlet with hypercharge  $Y = -1 - 1 = -2$ , while  $\bar{R}$  is an  $SU(2)_L$  singlet with hypercharge  $Y = +2$ . So we can write an invariant

$$\begin{aligned}\mathcal{L}_{\text{Yukawa}} &= -\lambda_e \bar{R}\phi^\dagger L + \text{c.c.} \\ &= -\lambda_e (\bar{R}\phi^\dagger L + \bar{L}\phi R)\end{aligned}$$

Here  $\lambda_e$  is the electron Yukawa coupling (not to be confused with the Higgs self-coupling  $\lambda$ ). If necessary we can redefine  $L$  and  $R$  by independent phases  $R \rightarrow e^{i\theta}R$ ,  $L \rightarrow e^{i\phi}L$  to make  $\lambda_e$  real and positive.

### 12.1.2 Mass spectrum and interactions

The model we've written down has an obvious phenomenological difficulty: the electron and all three  $W$  bosons seem to be massless.<sup>†</sup> Remarkably, the problem is cured by symmetry breaking. The Higgs potential is minimized when  $\phi^\dagger\phi = \mu^2/2\lambda$ . Under an  $SU(2)_L \times U(1)_Y$  gauge transformation we have

$$\phi \rightarrow e^{-ig\alpha^a(x)T_L^a}e^{-i\frac{g'}{2}\alpha(x)Y}\phi.$$

<sup>†</sup> Glashow, 1961: "It is a stumbling block we must overlook." I would have given up, he got the Nobel prize.

As far as the Higgs is concerned this is a  $U(2)$  transformation which can be used to put the expectation value into the standard form

$$\langle 0|\phi|0\rangle = \begin{pmatrix} 0 \\ v/\sqrt{2} \end{pmatrix} \quad (12.1)$$

where the Higgs vev  $v \equiv \sqrt{\mu^2/\lambda}$ . Now, what's the symmetry breaking pattern? We need to check which generators annihilate the vacuum:

$$\begin{aligned} \sigma^1 \langle \phi \rangle &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ v/\sqrt{2} \end{pmatrix} = \begin{pmatrix} v/\sqrt{2} \\ 0 \end{pmatrix} \neq 0 \\ \sigma^2 \langle \phi \rangle &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 0 \\ v/\sqrt{2} \end{pmatrix} = \begin{pmatrix} -iv/\sqrt{2} \\ 0 \end{pmatrix} \neq 0 \\ \sigma^3 \langle \phi \rangle &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ v/\sqrt{2} \end{pmatrix} = \begin{pmatrix} 0 \\ -v/\sqrt{2} \end{pmatrix} \neq 0 \\ Y \langle \phi \rangle &= \langle \phi \rangle = \begin{pmatrix} 0 \\ v/\sqrt{2} \end{pmatrix} \neq 0 \end{aligned}$$

It might look like the gauge group is completely broken, but in fact there's one linear combination of generators which leaves the vacuum invariant, namely  $Q \equiv T_L^3 + \frac{1}{2}Y$ .

$$\begin{aligned} \text{acting on } \phi: \quad Q &= \frac{1}{2}\sigma^3 + \frac{1}{2}(+1) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \\ \Rightarrow \quad Q \langle \phi \rangle &= 0 \end{aligned}$$

$Q$  generates an unbroken  $U(1)$  subgroup of  $SU(2)_L \times U(1)_Y$ , which we'll identify with the gauge group of electromagnetism. That is, we'll identify the eigenvalue of  $Q$  with electric charge.

To see that this makes sense, let's see how  $Q$  acts on our fields. We just have to keep in mind that  $T_L^3 = \frac{1}{2}\sigma^3$  when acting on a left-handed doublet, while  $T_L^3 = 0$  when acting on a singlet.

$$\begin{aligned} \text{acting on } L: \quad Q &= \frac{1}{2}\sigma^3 + \frac{1}{2}(-1) = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} & \text{matches } L = \begin{pmatrix} \nu \\ e \end{pmatrix}_L \\ \text{acting on } R: \quad Q &= 0 + \frac{1}{2}(-2) = -1 & \text{matches } R = e_R \\ \text{acting on } \phi: \quad Q &= \frac{1}{2}\sigma^3 + \frac{1}{2}(+1) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} & \text{matches } \phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix}_L \end{aligned}$$

A few comments are in order.

- (i) We got the expected electric charges for our fermions. This is no miracle: the hypercharge assignments were chosen to make this work.
- (ii) A gauge-invariant statement is that there is a negatively-charged left-handed spinor in the spectrum. Of course we identify this spinor with  $e_L$ . However the fact that  $e_L$  appears in the bottom component of a doublet is connected to our gauge choice (12.1). If we made a different gauge choice we'd have to change notation, as setting  $L = \begin{pmatrix} \nu \\ e \end{pmatrix}_L$  would no longer be appropriate.

What about the spectrum of masses? As usual, we expand about our choice of vacuum (12.1), setting<sup>†</sup>

$$\phi = \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}}(v + H(x)) \end{pmatrix}.$$

Here  $H(x)$  is a real scalar field, the physical Higgs field. When we plug this into the Yukawa Lagrangian we find

$$\begin{aligned} \mathcal{L}_{\text{Yukawa}} &= -\lambda_e (\bar{R}\phi^\dagger L + \bar{L}\phi R) \\ &= -\lambda_e \left[ \bar{e}_R \begin{pmatrix} 0 & \frac{1}{\sqrt{2}}(v + H) \end{pmatrix} \begin{pmatrix} \nu_L \\ e_L \end{pmatrix} + \begin{pmatrix} \bar{\nu}_L & \bar{e}_L \end{pmatrix} \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}}(v + H) \end{pmatrix} e_R \right] \\ &= -\frac{\lambda_e}{\sqrt{2}}(v + H)(\bar{e}_R e_L + \bar{e}_L e_R) \\ &= -\frac{\lambda_e}{\sqrt{2}}(v + H)\bar{e}e. \end{aligned}$$

In the last line we assembled  $e_L$  and  $e_R$  into a single Dirac spinor  $e$ . This gives the electron (but not the neutrino!) a mass,

$$m_e = \frac{\lambda_e v}{\sqrt{2}},$$

as well as a Yukawa coupling to the Higgs field:



<sup>†</sup> This is no loss of generality, as writing  $\phi$  in this way defines our choice of gauge for the broken symmetry generators. It's the standard model analog of the unitary gauge we adopted in section 11.1.

Next we look at the Higgs Lagrangian,

$$\mathcal{L}_{\text{Higgs}} = \mathcal{D}_\mu \phi^\dagger \mathcal{D}^\mu \phi + \mu^2 \phi^\dagger \phi - \lambda (\phi^\dagger \phi)^2$$

where

$$\begin{aligned} \mathcal{D}_\mu \phi &= \partial_\mu \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}}(v + H) \end{pmatrix} + \frac{ig}{2} \begin{pmatrix} W_\mu^3 & W_\mu^1 - iW_\mu^2 \\ W_\mu^1 + iW_\mu^2 & -W_\mu^3 \end{pmatrix} \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}}(v + H) \end{pmatrix} \\ &\quad + \frac{ig'}{2} B_\mu \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}}(v + H) \end{pmatrix} \\ &= \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{igv}{2} (W_\mu^1 - iW_\mu^2) \\ \partial_\mu H + \frac{iv}{2} (g' B_\mu - g W_\mu^3) \end{pmatrix} + (\text{terms quadratic in fields}) \end{aligned}$$

This means that

$$\begin{aligned} \mathcal{L}_{\text{Higgs}} &= \frac{1}{2} \partial_\mu H \partial^\mu H + \frac{1}{8} g^2 v^2 |W_\mu^1 - iW_\mu^2|^2 + \frac{1}{8} v^2 (g' B_\mu - g W_\mu^3)^2 - \mu^2 H^2 \\ &\quad + (\text{interaction terms}). \end{aligned}$$

Defining the linear combinations

$$\begin{aligned} W_\mu^\pm &= \frac{1}{\sqrt{2}} (W_\mu^1 \mp iW_\mu^2) \\ Z_\mu &= -\frac{g' B_\mu - g W_\mu^3}{\sqrt{g^2 + g'^2}} \\ A_\mu &= \frac{g B_\mu + g' W_\mu^3}{\sqrt{g^2 + g'^2}} \end{aligned}$$

we have

$$\mathcal{L}_{\text{Higgs}} = \frac{1}{2} \partial_\mu H \partial^\mu H - \mu^2 H^2 + \frac{1}{4} g^2 v^2 W_\mu^+ W^{\mu-} + \frac{1}{8} (g^2 + g'^2) v^2 Z_\mu Z^\mu + \text{interactions}$$

and we read off the masses

$$\begin{aligned} m_H^2 &= 2\mu^2 \\ m_W^2 &= \frac{1}{4} g^2 v^2 \\ m_Z^2 &= \frac{1}{4} (g^2 + g'^2) v^2 \\ m_A^2 &= 0 \end{aligned} \tag{12.2}$$

We have a massless photon (as required by the unbroken electromagnetic  $U(1)$ ), plus a massive Higgs scalar and a collection of massive intermediate vector bosons! Expanding  $\mathcal{L}_{\text{Higgs}}$  beyond quadratic order, one finds a slew

of interactions between the Higgs scalar and  $W$  and  $Z$  bosons; the Feynman rules are given in appendix E.

At this point it's convenient to introduce some standard notation. In place of the  $SU(2)_L \times U(1)_Y$  gauge couplings  $g, g'$  we'll often work in terms of the electromagnetic coupling  $e$  and the weak mixing angle  $\theta_W$ ,  $0 \leq \theta_W \leq \pi/2$ , defined by

$$\begin{aligned} e &= \frac{gg'}{\sqrt{g^2 + g'^2}} \\ \cos \theta_W &= g/\sqrt{g^2 + g'^2} \\ \sin \theta_W &= g'/\sqrt{g^2 + g'^2} \end{aligned}$$

I also like introducing the  $Z$  coupling, defined by

$$g_Z = \sqrt{g^2 + g'^2}.$$

In terms of these quantities note that

$$\begin{aligned} A_\mu &= \cos \theta_W B_\mu + \sin \theta_W W_\mu^3 \\ Z_\mu &= -\sin \theta_W B_\mu + \cos \theta_W W_\mu^3 \end{aligned} \tag{12.3}$$

while the  $Z$  mass is

$$m_Z^2 = \frac{1}{4} g_Z^2 v^2.$$

Now let's consider the Yang-Mills part of the action. Expanding in powers of the fields

$$\begin{aligned} \mathcal{L}_{\text{Yang-Mills}} &= -\frac{1}{2} \text{Tr} (W_{\mu\nu} W^{\mu\nu}) - \frac{1}{4} B_{\mu\nu} B^{\mu\nu} \\ &= -\frac{1}{4} W_{\mu\nu}^a W^{\mu\nu a} - \frac{1}{4} B_{\mu\nu} B^{\mu\nu} \\ &= -\frac{1}{4} (\partial_\mu W_\nu^a - \partial_\nu W_\mu^a) (\partial^\mu W^{\nu a} - \partial^\nu W^{\mu a}) - \frac{1}{4} (\partial_\mu B_\nu - \partial_\nu B_\mu) (\partial^\mu B^\nu - \partial^\nu B^\mu) \\ &\quad + (\text{interaction terms}) \end{aligned}$$

The quadratic terms have an  $SO(4)$  symmetry acting on  $(W_\mu^1, W_\mu^2, W_\mu^3, B_\mu)$ . So the  $SO(2)$  rotation (12.3) which mixes  $W_\mu^3$  with  $B_\mu$  just gives canonical kinetic terms for the fields  $W_\mu^\pm, Z_\mu, A_\mu$ . (We implicitly assumed this was the case when we read off the masses (12.2).) Expanding beyond quadratic order one finds a slew of gauge boson self-couplings: see appendix E for the Feynman rules.



Finally we consider the Dirac part of the action,

$$\mathcal{L}_{\text{Dirac}} = \bar{L} i \gamma^\mu \left( \partial_\mu + \frac{ig}{2} W_\mu^a \sigma^a + \frac{ig'}{2} B_\mu Y \right) L + \bar{R} i \gamma^\mu \left( \partial_\mu + \frac{ig'}{2} B_\mu Y \right) R.$$

This gives the fermions canonical kinetic terms, plus couplings to the gauge bosons

$$\mathcal{L}_{\text{Dirac}} = \cdots - g j_L^{\mu a} W_\mu^a - \frac{1}{2} g' j_Y^\mu B_\mu$$

where the  $SU(2)_L$  and hypercharge currents are

$$\begin{aligned} j_L^{\mu a} &= \bar{L} \gamma^\mu T^a L \quad \text{with} \quad T^a = \frac{1}{2} \sigma^a \\ j_Y^\mu &= \bar{L} \gamma^\mu Y L + \bar{R} \gamma^\mu Y R \end{aligned}$$

In terms of  $W^\pm = \frac{1}{\sqrt{2}} (W_\mu^1 \mp i W_\mu^2)$  this gives the charged-current couplings

$$\begin{aligned} \mathcal{L}_{\text{Dirac}}^{W^\pm} &= -g j_L^{\mu 1} W_\mu^1 - g j_L^{\mu 2} W_\mu^2 \\ &= -\frac{g}{\sqrt{2}} \left( (j_L^{\mu 1} + i j_L^{\mu 2}) W_\mu^+ + (j_L^{\mu 1} - i j_L^{\mu 2}) W_\mu^- \right) \\ &= -\frac{g}{\sqrt{2}} (\bar{L} \gamma^\mu \sigma^+ L W_\mu^+ + \bar{L} \gamma^\mu \sigma^- L W_\mu^-) \\ &= -\frac{g}{2\sqrt{2}} (\bar{\nu} \gamma^\mu (1 - \gamma^5) e W_\mu^+ + \bar{e} \gamma^\mu (1 - \gamma^5) \nu W_\mu^-) \end{aligned}$$

where in the third line  $\sigma^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ ,  $\sigma^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ . These are exactly the couplings that appeared in our old IVB amplitude (9.1)! Finally the couplings to  $\gamma$ ,  $Z$  are given by

$$\begin{aligned} \mathcal{L}_{\text{Dirac}}^{\gamma, Z} &= -g j_L^{\mu 3} W_\mu^3 - \frac{1}{2} g' j_Y^\mu B_\mu \\ &= -\left( g \cos \theta_W j_L^{\mu 3} - \frac{1}{2} g' \sin \theta_W j_Y^\mu \right) Z_\mu - \left( g \sin \theta_W j_L^{\mu 3} + \frac{1}{2} g' \cos \theta_W j_Y^\mu \right) A_\mu \\ &= -g_Z \left( \cos^2 \theta_W j_L^{\mu 3} - \frac{1}{2} \sin^2 \theta_W j_Y^\mu \right) Z_\mu - e \left( j_L^{\mu 3} + \frac{1}{2} j_Y^\mu \right) A_\mu \end{aligned}$$

It's convenient to eliminate the hypercharge current  $j_Y^\mu$  in favor of the electromagnetic current  $j_Q^\mu$ , using the definition  $j_Q^\mu = j_L^{\mu 3} + \frac{1}{2} j_Y^\mu$ . This leads to

$$\mathcal{L}_{\text{Dirac}}^{\gamma, Z} = -g_Z \left( j_L^{\mu 3} - \sin^2 \theta_W j_Q^\mu \right) Z_\mu - e j_Q^\mu A_\mu.$$

As advertised, the photon indeed couples to the vector-like<sup>†</sup> electromagnetic current with coupling constant  $e$ . Something like this was guaranteed to

<sup>†</sup> meaning the left- and right-handed part of the electron have the same electric charge

happen – the massless gauge field  $A_\mu$  must couple to the unbroken generator  $Q$ . One can write out the couplings a bit more explicitly in terms of a sum over fermions  $\psi_i$ ,  $i = \nu, e$ :

$$\begin{aligned}\mathcal{L}_{\text{Dirac}}^{\gamma, Z} &= -e \sum_i \bar{\psi}_i \gamma^\mu Q_i \psi_i A_\mu - g_Z \sum_i \bar{\psi}_i \gamma^\mu \left( \frac{1}{2} (1 - \gamma^5) T_{Li}^3 - \sin^2 \theta_W Q_i \right) \psi_i Z_\mu \\ &= -e \sum_i \bar{\psi}_i \gamma^\mu Q_i \psi_i A_\mu - \frac{1}{2} g_Z \sum_i \bar{\psi}_i (c_{Vi} \gamma^\mu - c_{Ai} \gamma^\mu \gamma^5) \psi_i Z_\mu\end{aligned}$$

where the vector and axial-vector couplings for each fermion are defined by

$$c_V = T_L^3 - 2 \sin^2 \theta_W Q \quad c_A = T_L^3.$$

Here  $T_L^3$  is the eigenvalue of  $\frac{1}{2} \sigma^3$  acting on the left-handed part of the field and  $Q$  is the electric charge of the field. So for example the electron has

$$\begin{aligned}c_{Ve} &= -\frac{1}{2} - 2 \sin^2 \theta_W (-1) = -\frac{1}{2} + 2 \sin^2 \theta_W \\ c_{Ae} &= -\frac{1}{2}\end{aligned}$$

while the neutrino has

$$c_{V\nu} = c_{A\nu} = \frac{1}{2}.$$

The Feynman rules from  $\mathcal{L}_{\text{Dirac}}$  can be found in appendix E.

### 12.1.3 Standard model parameters

Let's look at the parameters which appear in the standard model. With one generation of leptons there are only five parameters:

- Two gauge couplings  $g, g'$  (or equivalently the electric charge  $e = gg' / \sqrt{g^2 + g'^2}$  and the weak mixing angle  $\tan \theta_W = g'/g$ ).
- Two parameters in the Higgs potential  $\mu, \lambda$  (or equivalently the Higgs mass  $m_H = \sqrt{2}\mu$  and Higgs vev  $v = \sqrt{\mu^2/\lambda}$ ).
- The electron Yukawa coupling  $\lambda_e$  (or equivalently the electron mass  $m_e = \lambda_e v / \sqrt{2}$ ).

What do we know about the values of these parameters?

- The electric charge is known, of course:  $e^2/4\pi = 1/137$ .

- The weak mixing angle is determined by the ratio of the  $W$  and  $Z$  masses,

$$\sin^2 \theta_W = 1 - \cos^2 \theta_W = 1 - \frac{g^2}{g^2 + g'^2} = 1 - \frac{m_W^2}{m_Z^2} = 1 - \left( \frac{80.4 \text{ GeV}}{91.2 \text{ GeV}} \right)^2 = 0.223.$$

- To get the Higgs vev recall our old determination (9.3) of the Fermi constant in IVB theory,

$$\begin{aligned} G_F &= \frac{g^2}{4\sqrt{2}m_W^2} = \frac{1}{\sqrt{2}v^2} \\ \Rightarrow v &= \frac{1}{(\sqrt{2}G_F)^{1/2}} = 2^{-1/4}(1.17 \times 10^{-5} \text{ GeV}^{-2})^{-1/2} = 246 \text{ GeV}. \end{aligned}$$

It's kind of remarkable that the muon lifetime directly measures the Higgs vev.

- The electron mass determines the Yukawa coupling

$$\lambda_e = \frac{\sqrt{2}m_e}{v} = \frac{\sqrt{2} \times 0.511 \text{ MeV}}{246 \text{ GeV}} = 3 \times 10^{-6}.$$

One of the mysteries of the standard model is why the electron Yukawa is so small.

- The Higgs mass is the one parameter which has not been measured. Assuming the minimal standard model Higgs exists we only have limits on its mass. There's a lower limit

$$m_H > 114.4 \text{ GeV} \quad \text{at 95\% confidence}$$

from a direct search at LEP,<sup>†</sup> and an upper limit

$$m_H < 219 \text{ GeV} \quad \text{at 95\% confidence}$$

from a global fit to electroweak observables.<sup>‡</sup>

## 12.2 Electroweak interactions of quarks

To describe the electroweak interactions of a single generation of quarks the main challenge is to give a mass to the up quark. This is easier than one might have thought. We introduce left- and right-handed up and down quarks and assign them the  $SU(2)_L \times U(1)_Y$  quantum numbers

<sup>†</sup> hep-ex/0306033

<sup>‡</sup> hep-ex/0511027 p. 133. Also see table 10.2, but beware the large error bars.

$$\begin{array}{ll}
Q = \begin{pmatrix} u \\ d \end{pmatrix}_L & SU(2)_L \text{ doublet with hypercharge } Y = 1/3 \\
u_R & SU(2)_L \text{ singlet with hypercharge } Y = 4/3 \\
d_R & SU(2)_L \text{ singlet with hypercharge } Y = -2/3
\end{array}$$

The hypercharges are chosen so that  $Q = T_L^3 + \frac{1}{2}Y$  gives the quarks the correct electric charges. To break electroweak symmetry we have the Higgs doublet  $\phi$  with hypercharge +1. But we can also define

$$\tilde{\phi} = \epsilon \phi^*$$

where  $\epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ . Note that  $\tilde{\phi}$  is an  $SU(2)_L$  doublet with hypercharge  $-1$ .<sup>†</sup> This lets us build some invariants

$$\begin{array}{ll}
\phi d_R & SU(2)_L \text{ doublet with } Y = 1/3 \quad \Rightarrow \quad \bar{Q} \phi d_R \text{ invariant} \\
\tilde{\phi} u_R & SU(2)_L \text{ doublet with } Y = 1/3 \quad \Rightarrow \quad \bar{Q} \tilde{\phi} u_R \text{ invariant}
\end{array}$$

(There is no analog of the second invariant in the lepton sector, just because we didn't introduce a right-handed neutrino.) The general Yukawa Lagrangian is

$$\mathcal{L}_{\text{Yukawa}} = -\lambda_d \bar{Q} \phi d_R - \lambda_u \bar{Q} \tilde{\phi} u_R + \text{c.c.}$$

Here  $\lambda_d, \lambda_u$  are independent Yukawa couplings for the up and down quarks. Plugging in the Higgs vev this becomes

$$\begin{aligned}
\mathcal{L}_{\text{Yukawa}} &= -\lambda_d \begin{pmatrix} \bar{u}_L & \bar{d}_L \end{pmatrix} \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}}(v+H) \end{pmatrix} d_R - \lambda_u \begin{pmatrix} \bar{u}_L & \bar{d}_L \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}}(v+H) \\ 0 \end{pmatrix} u_R + \text{c.c.} \\
&= -\frac{1}{\sqrt{2}} \lambda_d (v+H) (\bar{d}_L d_R + \bar{d}_R d_L) - \frac{1}{\sqrt{2}} \lambda_u (v+H) (\bar{u}_L u_R + \bar{u}_R u_L) \\
&= -\frac{1}{\sqrt{2}} \lambda_d (v+H) \bar{d} d - \frac{1}{\sqrt{2}} \lambda_u (v+H) \bar{u} u
\end{aligned}$$

In the last line we assembled the chiral components  $u_L, u_R$  and  $d_L, d_R$  into Dirac spinors  $u$  and  $d$ . We read off the masses

$$m_u = \frac{\lambda_u v}{\sqrt{2}} \quad m_d = \frac{\lambda_d v}{\sqrt{2}}.$$

There's also a Yukawa coupling to the Higgs; the Feynman rule is in appendix E. In a way we're fortunate here – the hypercharge assignments are such that the same Higgs doublet which gives a mass to the electron can also be used to give a mass to the up and down quarks.

<sup>†</sup> In  $SU(2)$  index notation  $\tilde{\phi}^i = \epsilon^{ij} \phi_j = \epsilon^{ij} (\phi^j)^*$ . The hypercharge changes sign due to the complex conjugation.

### 12.3 Multiple generations

Now let's write the standard model with three generations of quarks and leptons. It's basically a matter of sprinkling generation indices  $i, j = 1, 2, 3$  on our quark and lepton fields. Including color for completeness, the gauge quantum numbers are

<i>field</i>	$SU(3)_C \times SU(2)_L \times U(1)_Y$ <i>quantum numbers</i>
left-handed leptons $L_i = \begin{pmatrix} \nu_{Li} \\ e_{Li} \end{pmatrix}$	$(\mathbf{1}, \mathbf{2}, -1)$
right-handed leptons $e_{Ri}$	$(\mathbf{1}, \mathbf{1}, -2)$
left-handed quarks $Q_i = \begin{pmatrix} u_{Li} \\ d_{Li} \end{pmatrix}$	$(\mathbf{3}, \mathbf{2}, 1/3)$
right-handed up-type quarks $u_{Ri}$	$(\mathbf{3}, \mathbf{1}, 4/3)$
right-handed down-type quarks $d_{Ri}$	$(\mathbf{3}, \mathbf{1}, -2/3)$

The standard model Lagrangian is written in appendix E. The main new wrinkle is that the Yukawa couplings get promoted to  $3 \times 3$  complex matrices  $\Lambda_{ij}^e, \Lambda_{ij}^d, \Lambda_{ij}^u$ .

$$\begin{aligned}
 \mathcal{L}_{\text{Yukawa}} &= -\Lambda_{ij}^e \bar{L}_i \phi e_{Rj} - \Lambda_{ij}^d \bar{Q}_i \phi d_{Rj} - \Lambda_{ij}^u \bar{Q}_i \tilde{\phi} u_{Rj} + \text{c.c.} \\
 &= -\bar{L} \Lambda^e \phi e - \bar{Q} \Lambda^d \phi d - \bar{Q} \Lambda^u \tilde{\phi} u + \text{c.c.}
 \end{aligned}$$

In the second line we adopted matrix notation and suppressed the generation indices as well as the subscripts  $R$  on the right-handed fields.

We'd like to diagonalize the fermion mass matrices. To do this we use the fact that a general complex matrix can be diagonalized by a bi-unitary transformation,

$$\Lambda = U_L \lambda U_R^\dagger$$

where  $U_L$  and  $U_R$  are unitary and  $\lambda$  is a diagonal matrix with entries that are real and non-negative.<sup>†</sup> Then

$$\mathcal{L}_{\text{Yukawa}} = -\bar{L} U_L^e \lambda_e U_R^{e\dagger} \phi e - \bar{Q} U_L^d \lambda_d U_R^{d\dagger} \phi d - \bar{Q} U_L^u \lambda_u U_R^{u\dagger} \tilde{\phi} u + \text{c.c.}$$

Now let's redefine our fermion fields

$$L \rightarrow U_L^e L$$

<sup>†</sup> Proof:  $\Lambda^\dagger \Lambda$  is Hermitian with non-negative eigenvalues, so  $\Lambda^\dagger \Lambda = U_R \lambda^2 U_R^\dagger$  for some unitary matrix  $U_R$  and some diagonal, real, non-negative matrix  $\lambda$ . Then  $\Lambda^\dagger \Lambda = (\lambda U_R^\dagger)^\dagger (\lambda U_R^\dagger) \Rightarrow \Lambda = U_L \lambda U_R^\dagger$  for some unitary  $U_L$ .

$$\begin{aligned}
e &\rightarrow U_R^e e \\
Q &= \begin{pmatrix} u_L \\ d_L \end{pmatrix} \rightarrow \begin{pmatrix} U_L^u & 0 \\ 0 & U_L^d \end{pmatrix} Q \\
d &\rightarrow U_R^d d \\
u &\rightarrow U_R^u u
\end{aligned}$$

Note that we're transforming the up-type and down-type components of  $Q$  differently. Keeping in mind that with our gauge choice only one component of the Higgs doublet is non-zero, this transformation makes the Yukawa Lagrangian flavor-diagonal.

$$\mathcal{L}_{\text{Yukawa}} \rightarrow -\bar{L}\lambda_e\phi e - \bar{Q}\lambda_d\phi d - \bar{Q}\lambda_u\tilde{\phi}u + \text{c.c.}$$

So in terms of these redefined fermions we have diagonal mass matrices and flavor-diagonal couplings to the Higgs. What happens to the rest of the standard model Lagrangian? The transformation doesn't affect the Higgs or Yang-Mills sectors, of course. And the Dirac Lagrangian

$$\mathcal{L}_{\text{Dirac}} = \bar{L}i\gamma^\mu\mathcal{D}_\mu L + \bar{e}i\gamma^\mu\mathcal{D}_\mu e + \bar{Q}i\gamma^\mu\mathcal{D}_\mu Q + \bar{u}i\gamma^\mu\mathcal{D}_\mu u + \bar{d}i\gamma^\mu\mathcal{D}_\mu d$$

is invariant when  $L, e, u, d$  are multiplied by unitary matrices, so terms involving those fields aren't affected. The only terms in  $\mathcal{L}_{\text{Dirac}}$  that are affected involve  $Q$ . Writing out the  $SU(2)_L$  part of the covariant derivative explicitly

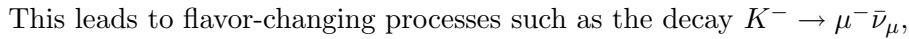
$$\begin{aligned}
\mathcal{L}_{\text{Dirac}} &= \dots + \bar{Q}i\gamma^\mu\mathcal{D}_\mu Q \\
&\rightarrow \dots + \bar{Q}i\gamma^\mu \begin{pmatrix} U_L^{u\dagger} & 0 \\ 0 & U_L^{d\dagger} \end{pmatrix} \left[ \partial_\mu + ig_s G_\mu + \frac{ig}{2} \begin{pmatrix} W_\mu^3 & W_\mu^1 - iW_\mu^2 \\ W_\mu^1 + iW_\mu^2 & -W_\mu^3 \end{pmatrix} + \frac{ig'}{2} B_\mu Y \right] \begin{pmatrix} U_L^u & 0 \\ 0 & U_L^d \end{pmatrix} Q \\
&= \dots + \bar{Q}i\gamma^\mu \left[ \partial_\mu + ig_s G_\mu + \frac{ig}{2} \begin{pmatrix} W_\mu^3 & (W_\mu^1 - iW_\mu^2)U_L^{u\dagger}U_L^d \\ (W_\mu^1 + iW_\mu^2)U_L^{d\dagger}U_L^u & -W_\mu^3 \end{pmatrix} + \frac{ig'}{2} B_\mu Y \right] Q
\end{aligned}$$

So in fact the only place the transformation shows up is in the quark – quark –  $W^\pm$  couplings.

$$\begin{aligned}
\mathcal{L}_{\text{Dirac}}^{qqW^\pm} &= -\frac{g}{2} \begin{pmatrix} \bar{u}_L & \bar{d}_L \end{pmatrix} \gamma^\mu \begin{pmatrix} 0 & (W_\mu^1 - iW_\mu^2)U_L^{u\dagger}U_L^d \\ (W_\mu^1 + iW_\mu^2)U_L^{d\dagger}U_L^u & 0 \end{pmatrix} \begin{pmatrix} u_L \\ d_L \end{pmatrix} \\
&= -\frac{g}{2\sqrt{2}} \bar{u}\gamma^\mu(1 - \gamma^5)W_\mu^+ V d - \frac{g}{2\sqrt{2}} \bar{d}\gamma^\mu(1 - \gamma^5)W_\mu^- V^\dagger u
\end{aligned}$$

Here  $V \equiv U_L^{u\dagger}U_L^d$  is the CKM matrix. It's a  $3 \times 3$  unitary matrix that governs intergenerational mixing in charged-current weak interactions.<sup>†</sup> The Feynman rules are

<sup>†</sup> One of the peculiar things about the standard model is that – for no particularly good reason – the weak neutral current is flavor-diagonal.

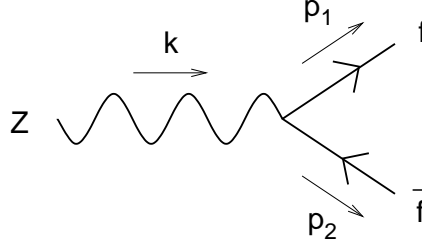


## 12.4 Some sample calculations

We'll conclude by discussing a few calculations in the standard model: decay of the  $Z$ ,  $e^+e^-$  annihilation near the  $Z$  pole, Higgs production and decay.

### 12.4.1 Decay of the $Z$

The  $Z$  decays to a fermion – antifermion pair, via the tree-level diagram



The amplitude is easy to write down.

$$-i\mathcal{M} = \bar{u}(p_1) \left( -\frac{ig_Z}{2} \gamma^\mu (c_V - c_A \gamma^5) \right) v(p_2) \epsilon_\mu$$

Summing over the spins in the final state, and neglecting all fermion masses for simplicity, we have

$$\begin{aligned} \sum_{\text{final spins}} |\mathcal{M}|^2 &= \frac{1}{4} g_Z^2 \text{Tr} \left( \gamma^\mu (c_V - c_A \gamma^5) \not{p}_2 \gamma^\nu (c_V - c_A \gamma^5) \not{p}_1 \right) \epsilon_\mu \epsilon_\nu^* \\ &= \frac{1}{4} g_Z^2 \text{Tr} \left( \gamma^\mu \not{p}_2 \gamma^\nu \not{p}_1 (c_V^2 + c_A^2 + 2c_V c_A \gamma^5) \right) \epsilon_\mu \epsilon_\nu^* \end{aligned}$$

We now average over  $Z$  polarizations, using  $\sum \epsilon_\mu \epsilon_\nu^* = -g_{\mu\nu} + \frac{k_\mu k_\nu}{m_Z^2}$ , and evaluate the Dirac traces, using

$$\begin{aligned} \text{Tr} \left( \gamma^\mu \gamma^\lambda \gamma^\nu \gamma^\sigma \right) &= 4 \left( g^{\mu\lambda} g^{\nu\sigma} - g^{\mu\nu} g^{\lambda\sigma} + g^{\mu\sigma} g^{\lambda\nu} \right) \\ \text{Tr} \left( \gamma^\mu \gamma^\lambda \gamma^\nu \gamma^\sigma \gamma^5 \right) &= 4i \epsilon^{\mu\lambda\nu\sigma} \end{aligned}$$

Traces involving  $\gamma^5$  drop out since they're antisymmetric on  $\mu$  and  $\nu$ . We're left with

$$\begin{aligned} \langle |\mathcal{M}|^2 \rangle &= \frac{1}{12} g_Z^2 (c_V^2 + c_A^2) 4(p_2^\mu p_1^\nu - g^{\mu\nu} p_1 \cdot p_2 + p_1^\mu p_2^\nu) \left( -g_{\mu\nu} + \frac{k_\mu k_\nu}{m_Z^2} \right) \\ &= \frac{1}{3} g_Z^2 (c_V^2 + c_A^2) \left( p_1 \cdot p_2 + \frac{2k \cdot p_1 k \cdot p_2}{m_Z^2} \right) \end{aligned}$$

where we used  $k^2 = m_Z^2$ . In the rest frame of the  $Z$  we have

$$\begin{aligned} k &= (m_Z, 0, 0, 0) & p_1 &= \left( \frac{m_Z}{2}, 0, 0, \frac{m_Z}{2} \right) & p_2 &= \left( \frac{m_Z}{2}, 0, 0, -\frac{m_Z}{2} \right) \\ \Rightarrow k \cdot p_1 &= k \cdot p_2 = p_1 \cdot p_2 & &= \frac{m_Z^2}{2} \end{aligned}$$



and the amplitude is simply

$$\langle |\mathcal{M}|^2 \rangle = \frac{1}{3} g_Z^2 (c_V^2 + c_A^2) m_Z^2.$$

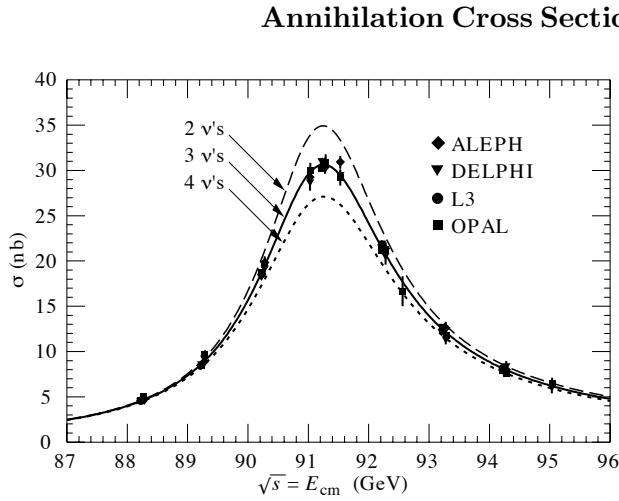
The partial width for the decay  $Z \rightarrow f \bar{f}$  is then

$$\Gamma_{Z \rightarrow f \bar{f}} = \frac{|\mathbf{p}|}{8\pi m_Z^2} \langle |\mathcal{M}|^2 \rangle = \frac{1}{3} \alpha_Z m_Z (c_V^2 + c_A^2)$$

where we've introduced the  $Z$  analog of the fine structure constant  $\alpha_Z = \frac{(g_Z/2)^2}{4\pi} = 1/91$ . Summing over fermions we find the total width of the  $Z$

$$\begin{aligned} \Gamma_Z &= \frac{1}{3} \alpha_Z m_Z \sum_f (c_{Vf}^2 + c_{Af}^2) \\ &= 0.334 \text{ GeV} \left[ \underbrace{3 \times 0.50}_{\nu_e \nu_\mu \nu_\tau} + \underbrace{3 \times 0.251}_{e \mu \tau} + \underbrace{6 \times 0.287}_{uc} + \underbrace{9 \times 0.370}_{dsb} \right] \end{aligned}$$

where the contributions of the various fermions are indicated (don't forget to sum over quark colors!). This gives a total width  $\Gamma_Z = 2.44 \text{ GeV}$ , not bad compared to the observed value  $\Gamma_{\text{obs}} = 2.50 \text{ GeV}$ . The “invisible width” of the  $Z$  can be inferred quite accurately, since (as we'll discuss) the total width shows up in the cross section for  $e^+e^- \rightarrow \text{hadrons}$  near the  $Z$  pole. In the standard model the invisible width comes from decays to neutrinos which escape the detector. Knowing the invisible width allows us to count the number of neutrino species  $N_\nu$  which couple to the  $Z$  and have masses less than  $m_Z/2$ . The particle data group gives  $N_\nu = 2.92 \pm 0.07$ .



**Figure 39.9:** Data from the ALEPH, DELPHI, L3, and OPAL Collaborations for the cross section in  $e^+e^-$  annihilation into hadronic final states as a function of c.m. energy near the  $Z$ . LEP detectors obtained data at the same energies; some of the points are obscured by overlap. The curves show the predictions of the Standard Model with three species (solid curve) and four species (dashed curve) of light neutrinos. The asymmetry of the curves is produced by initial-state radiation. References:

**ALEPH:** D. Decamp *et al.*, Z. Phys. **C53**, 1 (1992).

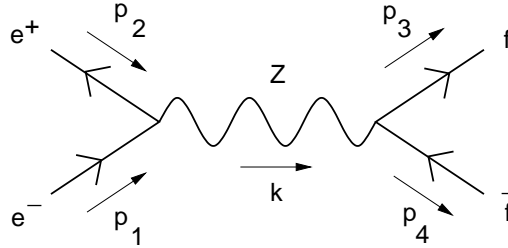
**DELPHI:** P. Abreu *et al.*, Nucl. Phys. **B367**, 511 (1992).

**L3:** B. Adeva *et al.*, Z. Phys. **C51**, 179 (1991).

**OPAL:** G. Alexander *et al.*, Z. Phys. **C52**, 175 (1991).

#### 12.4.2 $e^+e^-$ annihilation near the $Z$ pole

At energies near  $m_Z$  the process  $e^+e^- \rightarrow f\bar{f}$  is dominated by the formation of an intermediate  $Z$  resonance.



The amplitude for this process is

$$\begin{aligned}
 -i\mathcal{M} = & \bar{v}(p_2) \left( -\frac{ig_Z}{2} (c_{Ve}\gamma^\mu - c_{Ae}\gamma^\mu\gamma^5) \right) u(p_1) \frac{-i \left( g_{\mu\nu} - \frac{k_\mu k_\nu}{m_Z^2} \right)}{k^2 - m_Z^2} \\
 & \bar{u}(p_3) \left( -\frac{ig_Z}{2} (c_{Vf}\gamma^\nu - c_{Af}\gamma^\nu\gamma^5) \right) v(p_4)
 \end{aligned}$$

For simplicity let's neglect the external fermion masses. Then, just as in our calculation of inverse muon decay in section 9.3, the  $k_\mu k_\nu / m_Z^2$  term in the

$Z$  propagator can be neglected. To see this note that  $k = p_1 + p_2 = p_3 + p_4$  and use the Dirac equation for the external lines. This leaves

$$-i\mathcal{M} = \frac{ig_Z^2}{4(k^2 - m_Z^2)} \bar{v}(p_2)(c_{Ve}\gamma^\mu - c_{Ae}\gamma^\mu\gamma^5)u(p_1) \bar{u}(p_3)(c_{Vf}\gamma_\mu - c_{Af}\gamma_\mu\gamma^5)v(p_4).$$

It's convenient to work in terms of chiral spinors. Suppose all the spinors appearing in our amplitude are right-handed. Recalling the connection between chirality and helicity for massless fermions, this means the amplitude for polarized scattering  $e_L^+ e_R^- \rightarrow f_R \bar{f}_L$  is

$$-i\mathcal{M}_{e_L^+ e_R^- \rightarrow f_R \bar{f}_L} = \frac{ig_Z^2}{4(k^2 - m_Z^2)} (c_{Ve} - c_{Ae})(c_{Vf} - c_{Af}) \bar{v}_L(p_2)\gamma^\mu u_R(p_1) \bar{u}_R(p_3)\gamma_\mu v_L(p_4)$$

where the subscripts  $L, R$  indicate particle helicities. Using the explicit form of the spinors given in section 4.1 we have

$$\bar{v}_L(p_2)\gamma^\mu u_R(p_1) \bar{u}_R(p_3)\gamma_\mu v_L(p_4) = -k^2(1 + \cos\theta)$$

where  $\theta$  is the center of mass scattering angle. (We worked out this angular dependence in section 4.2. Here we're keeping track of the normalization as well.) This means

$$|\mathcal{M}|_{e_L^+ e_R^- \rightarrow f_R \bar{f}_L}^2 = \frac{g_Z^4 k^4}{16(k^2 - m_Z^2)^2} (c_{Ve} - c_{Ae})^2 (c_{Vf} - c_{Af})^2 (1 + \cos\theta)^2.$$

At this point we need to take the finite lifetime of the  $Z$  into account. As usual in quantum mechanics we can regard the width of an unstable state as an imaginary contribution to its energy, so we can take the width of the  $Z$  into account by replacing  $m_Z \rightarrow m_Z - i\Gamma_Z/2$ . This modifies the  $Z$  propagator,

$$\frac{1}{k^2 - m_Z^2} \rightarrow \frac{1}{k^2 - (m_Z - i\Gamma_Z/2)^2} \approx \frac{1}{k^2 - m_Z^2 + im_Z\Gamma_Z}$$

where we assumed the width was small compared to the mass. With this modification

$$|\mathcal{M}|_{e_L^+ e_R^- \rightarrow f_R \bar{f}_L}^2 = \frac{g_Z^4 k^4}{16((k^2 - m_Z^2)^2 + m_Z^2\Gamma_Z^2)} (c_{Ve} - c_{Ae})^2 (c_{Vf} - c_{Af})^2 (1 + \cos\theta)^2.$$

Now we can work at resonance and set  $k^2 = m_Z^2$  to find

$$|\mathcal{M}|_{e_L^+ e_R^- \rightarrow f_R \bar{f}_L}^2 = \frac{g_Z^4 m_Z^2}{16\Gamma_Z^2} (c_{Ve} - c_{Ae})^2 (c_{Vf} - c_{Af})^2 (1 + \cos\theta)^2.$$

Plugging this into  $\frac{d\sigma}{d\Omega} = \frac{|\mathcal{M}|^2}{64\pi^2 s}$  and integrating over angles gives the cross section for polarized scattering at the  $Z$  pole.

$$\sigma_{e_L^+ e_R^- \rightarrow f_R \bar{f}_L} = \frac{4\pi\alpha_Z^2}{3\Gamma_Z^2} (c_{Ve} - c_{Ae})^2 (c_{Vf} - c_{Af})^2$$

The other polarized cross sections are almost identical, one just gets  $\pm$  signs depending on the spinor chiralities.

$$\begin{aligned}\sigma_{e_L^+ e_R^- \rightarrow f_L \bar{f}_R} &= \frac{4\pi\alpha_Z^2}{3\Gamma_Z^2} (c_{Ve} - c_{Ae})^2 (c_{Vf} + c_{Af})^2 \\ \sigma_{e_R^+ e_L^- \rightarrow f_R \bar{f}_L} &= \frac{4\pi\alpha_Z^2}{3\Gamma_Z^2} (c_{Ve} + c_{Ae})^2 (c_{Vf} - c_{Af})^2 \\ \sigma_{e_R^+ e_L^- \rightarrow f_L \bar{f}_R} &= \frac{4\pi\alpha_Z^2}{3\Gamma_Z^2} (c_{Ve} + c_{Ae})^2 (c_{Vf} + c_{Af})^2\end{aligned}$$

Averaging over initial spins and summing over final spins, the unpolarized cross section is

$$\sigma = \frac{4\pi\alpha_Z^2}{3\Gamma_Z^2} (c_{Ve}^2 + c_{Ae}^2) (c_{Vf}^2 + c_{Af}^2).$$

Now we can compute the cross section for  $e^+e^- \rightarrow \text{hadrons}$  by summing over  $f = u, c, d, s, b$ . Using our result for the  $Z$  width we'd estimate

$$\sigma(e^+e^- \rightarrow \text{hadrons}) = 1.1 \times 10^{-4} \text{ GeV}^{-2} = 43 \text{ nb}$$

which isn't bad compared to the PDG value  $\sigma_{\text{had}} = 41.5 \text{ nb}$ . We can also estimate the cross section ratio

$$R = \frac{\sigma(e^+e^- \rightarrow \text{hadrons})}{\sigma(e^+e^- \rightarrow \mu^+\mu^-)}.$$

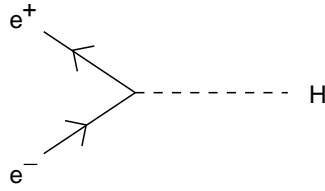
Recalling that the QED cross section is  $\sigma(e^+e^- \rightarrow \mu^+\mu^-) = 4\pi\alpha^2/3s$  we'd estimate that near the  $Z$  pole

$$R = \frac{\alpha_Z^2 m_Z^2}{\alpha^2 \Gamma_Z^2} (c_{Ve}^2 + c_{Ae}^2) \sum_{f=u,c,d,s,b} (c_{Vf}^2 + c_{Af}^2) \approx 3510$$

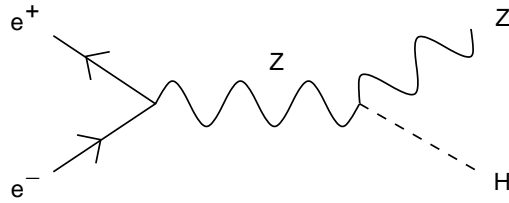
which again is pretty close to the observed value (see the plot in chapter 3).

### 12.4.3 Higgs production and decay

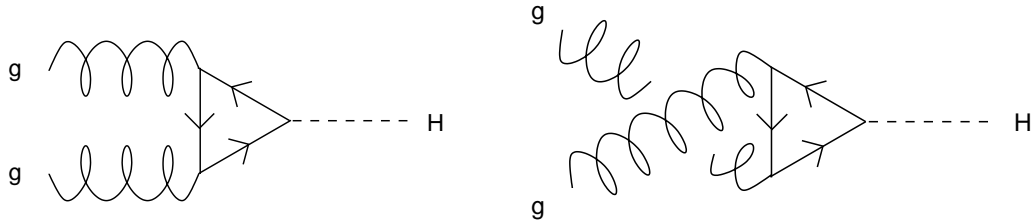
Finally, a few words on Higgs production and decay. At an  $e^+e^-$  collider the simplest production mechanism



is negligible due to the small electron Yukawa coupling. The dominant production mechanism is  $e^+e^- \rightarrow Z^* \rightarrow ZH$  via the diagram

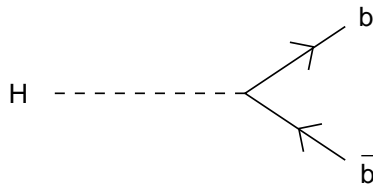


At a hadron collider the main production mechanism is “gluon fusion,” in which two gluons make a Higgs via a quark loop.

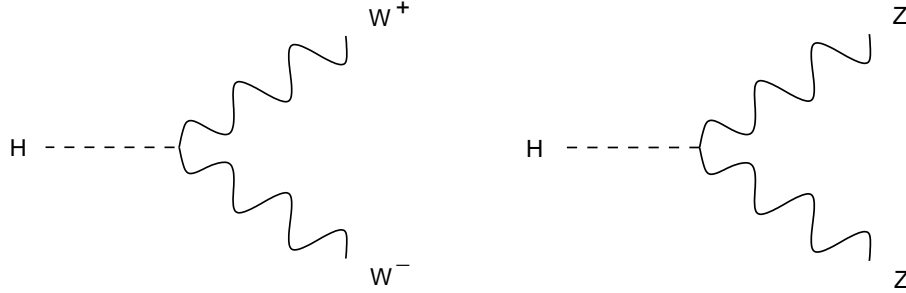


The biggest contribution comes from a loop of top quarks: the enhancement of the diagram due to the large top Yukawa coupling turns out to win over the suppression due to the large top mass.

For a light Higgs (meaning  $m_H < 140 \text{ GeV}$ ) the most important decay is  $H \rightarrow b\bar{b}$ ; the  $b$  quark is favored since it has the largest Yukawa coupling.



For a heavy Higgs (meaning  $m_H > 140 \text{ GeV}$ ) the decays  $H \rightarrow W^+W^-$  and  $H \rightarrow ZZ$  become possible and turn out to be the dominant decay modes.



(If  $m_H < 2m_W$  or  $m_H < 2m_Z$  one of the vector bosons is off-shell.) The nature of the Higgs depends on its mass. A light Higgs is a quite narrow resonance, but the Higgs width increases rapidly above the  $W^+W^-$  threshold.

What might we hope to see at the LHC? For a light Higgs the dominant  $b\bar{b}$  decay mode is obscured by QCD backgrounds and one has to look for rare decays. A leading candidate is  $H \rightarrow \gamma\gamma$  which can occur through a top quark triangle (similar to the gluon fusion diagrams drawn above) or through a  $W$  loop. Somewhat counter-intuitively it's easier to find a heavy Higgs. If  $m_H > 2m_Z$  there are clean signals available, most notably  $H \rightarrow ZZ \rightarrow \mu^+\mu^-\mu^+\mu^-$ .

## References

There's a nice development of the standard model in Quigg, section 6.3 for leptons and section 7.1 for quarks. Cheng & Li covers the standard model in section 11.2 and treats quark mixing in section 11.3. Higgs physics and electroweak symmetry breaking is reviewed in S. Dawson, *Introduction to electroweak symmetry breaking*, hep-ph/9901280 and in L. Reina, *TASI 2004 lecture notes on Higgs boson physics*, hep-ph/0512377. The standard model is summarized in appendix E of these notes.

From Reina, p. 35:

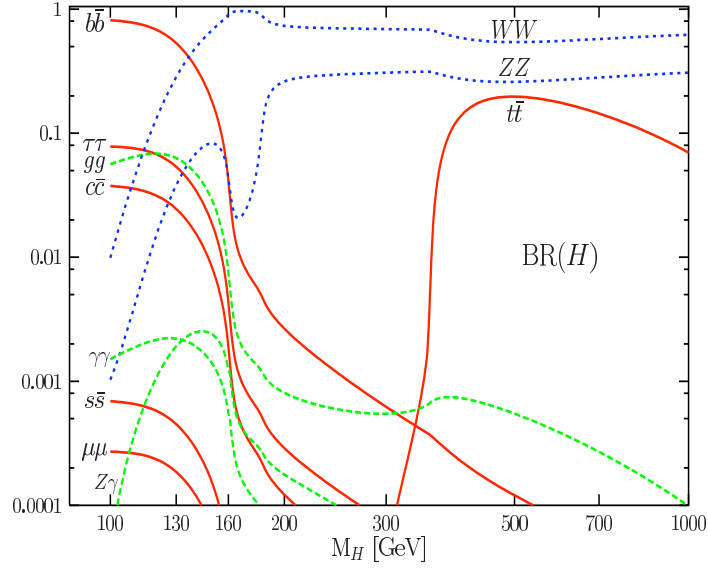


FIG. 9: SM Higgs decay branching ratios as a function of  $M_H$ . The blue curves represent tree-level decays into electroweak gauge bosons, the red curves tree level decays into quarks and leptons, the green curves one-loop decays. From Ref. [6].

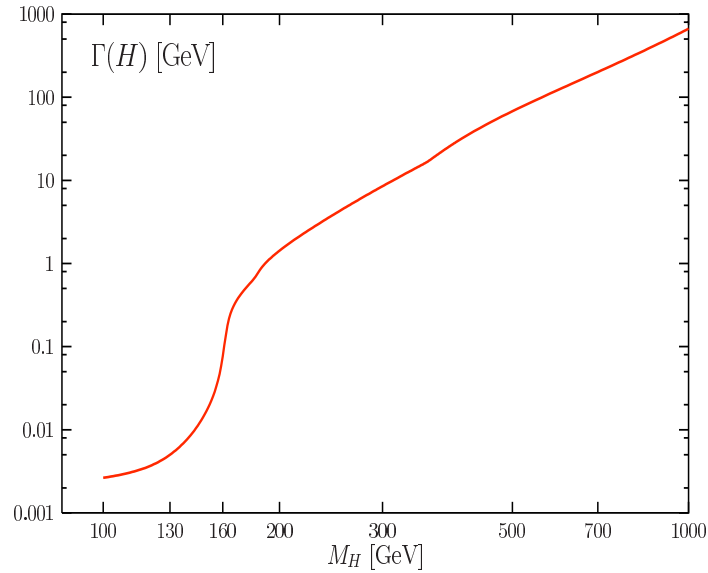
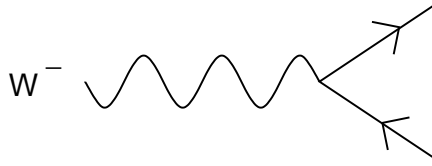


FIG. 10: SM Higgs total decay width as a function of  $M_H$ . From Ref. [6].

## Exercises

12.1  **$W$  decay**

The  $W^-$  can decay to a weak doublet pair of fermions via the diagram



Use this to compute the total width of the  $W$ . How did you do compared to the observed value  $2.085 \pm 0.042 \text{ GeV}$ ? A few hints:

- aside from the top quark, it's okay to neglect fermion masses
- see if you can write your answer in terms of  $\alpha$  and  $\sin^2 \theta_W$ , where at the scale  $m_W$  these quantities have the values

$$\alpha = 1/128 \quad (\text{not } 1/137!)$$

$$\sin^2 \theta_W = 0.231$$

- when summing over quarks in the final state, it helps to remember that the CKM matrix is unitary,  $(VV^\dagger)_{ij} = \delta_{ij}$

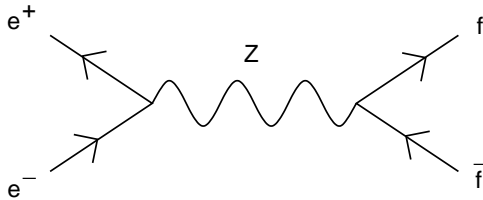
12.2 **Polarization asymmetry at the  $Z$  pole**

SLAC studied  $e^-e^+ \rightarrow f\bar{f}$  at the  $Z$  pole with a polarized  $e^-$  beam. The polarization asymmetry is defined by

$$A_{LR} = \frac{\sigma(e_L^- e_R^+ \rightarrow f\bar{f}) - \sigma(e_R^- e_L^+ \rightarrow f\bar{f})}{\sigma(e_L^- e_R^+ \rightarrow f\bar{f}) + \sigma(e_R^- e_L^+ \rightarrow f\bar{f})}$$

where the subscripts indicate the helicity of the particles. For simplicity you can neglect the mass of the electron, but you should keep  $m_f \neq 0$ .

- (i) Write down the amplitude for the basic process

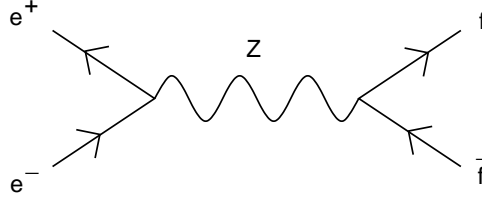




- (ii) Write down the amplitude when the incoming electron is polarized, either left-handed  $\gamma^5 u(p_1) = -u(p_1)$  or right-handed  $\gamma^5 u(p_1) = +u(p_1)$ .
- (iii) Compute  $A_{LR}$  in terms of  $\sin^2 \theta_W$ . You can do this without using any trace theorems!
- (iv) The observed asymmetry in  $e^- e^+ \rightarrow \text{hadrons}$  is  $A_{LR} = 0.1514 \pm 0.002$ . How well did you do?

### 12.3 Forward-backward asymmetries at the $Z$ pole

Consider unpolarized scattering  $e^+ e^- \rightarrow f \bar{f}$  near the  $Z$  pole. The diagram is



The forward-backward asymmetry  $A_{FB}^f$  is defined in terms of the cross sections for forward and backward scattering by

$$\begin{aligned}\sigma_F &= 2\pi \int_0^1 d(\cos \theta) \left( \frac{d\sigma}{d\Omega} \right)_{e^+ e^- \rightarrow f \bar{f}} \\ \sigma_B &= 2\pi \int_{-1}^0 d(\cos \theta) \left( \frac{d\sigma}{d\Omega} \right)_{e^+ e^- \rightarrow f \bar{f}} \\ A_{FB}^f &= \frac{\sigma_F - \sigma_B}{\sigma_F + \sigma_B}\end{aligned}$$

Here  $\theta$  is the scattering angle measured in the center of mass frame between the outgoing fermion  $f$  and the incoming positron beam.

- (i) Write down the differential cross sections for the *polarized* processes

$$\begin{aligned}e_L^+ e_R^- &\rightarrow f_L \bar{f}_R \\ e_L^+ e_R^- &\rightarrow f_R \bar{f}_L \\ e_R^+ e_L^- &\rightarrow f_L \bar{f}_R \\ e_R^+ e_L^- &\rightarrow f_R \bar{f}_L\end{aligned}$$

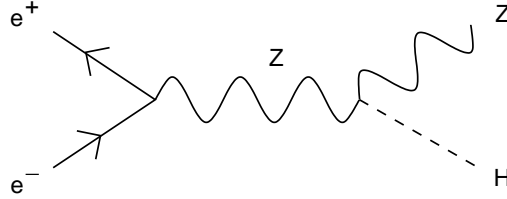
You can neglect the masses of the external particles. Also you don't need to keep track of any overall normalizations that would

end up cancelling out of  $A_{FB}^f$ . Hint: rather than use trace theorems, you should use results from our discussion of polarized scattering  $e^+e^- \rightarrow \mu^+\mu^-$  in QED.

- (ii) Compute  $A_{FB}^f$  for  $f = e, \mu, \tau, b, c, s$ . How well did you do, compared to the particle data book? (See table 10.4 in the section “Electroweak model and constraints on new physics,” where  $A_{FB}^f$  is denoted  $A_{FB}^{(0,f)}$ .)

#### 12.4 $e^+e^- \rightarrow ZH$

Compute the cross section for  $e^+e^- \rightarrow ZH$  from the diagram



For simplicity you can neglect the electron mass. You should find

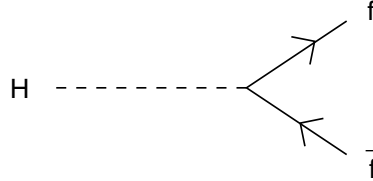
$$\sigma = \frac{\pi\alpha_Z^2\lambda^{1/2}(\lambda + 12m_Z^2/s)}{12s(1 - m_Z^2/s)^2} \left(1 + (1 - 4\sin^2\theta_W)^2\right)$$

where

$$\lambda = \left(1 - \frac{m_Z^2 + m_H^2}{s}\right)^2 - \frac{4m_Z^2m_H^2}{s^2}.$$

#### 12.5 $H \rightarrow f\bar{f}, W^+W^-, ZZ$

- (i) Compute the partial width for the decay  $H \rightarrow f\bar{f}$  from the diagram

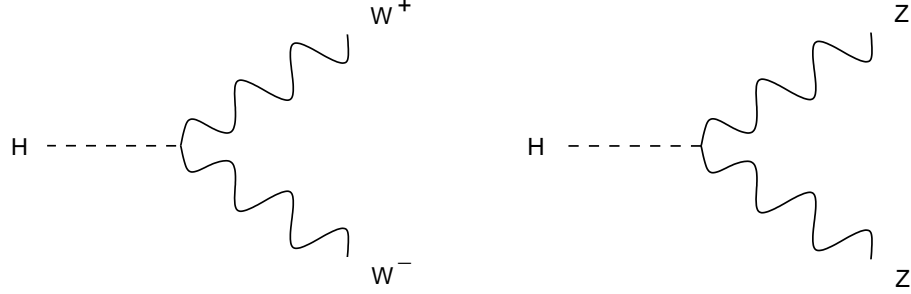


For leptons you should find

$$\Gamma(H \rightarrow f\bar{f}) = \frac{m_f^2}{8\pi m_H^2 v^2} (m_H^2 - 4m_f^2)^{3/2}$$

while for quarks the color sum enhances the width by a factor of three.

- (ii) Compute the partial widths for the decays  $H \rightarrow W^+W^-$  and  $H \rightarrow ZZ$  from the diagrams



The Feynman rules are in appendix E. You should find

$$\Gamma(H \rightarrow W^+W^-) = \frac{m_H^3}{16\pi v^2} (1 - r_W)^{1/2} \left( 1 - r_W + \frac{3}{4} r_W^2 \right)$$

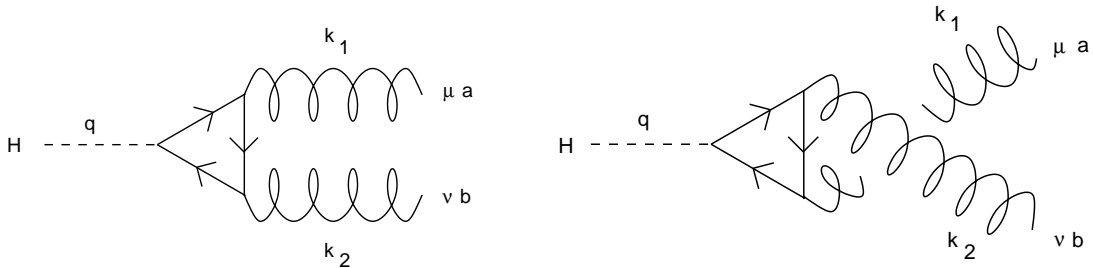
$$\Gamma(H \rightarrow ZZ) = \frac{m_H^3}{32\pi v^2} (1 - r_Z)^{1/2} \left( 1 - r_Z + \frac{3}{4} r_Z^2 \right)$$

where  $r_W = 4m_W^2/m_H^2$  and  $r_Z = 4m_Z^2/m_H^2$ .

- (iii) Show that a heavy Higgs particle will decay predominantly to longitudinally-polarized vector bosons. That is, show that for large  $m_H$  the total width of the Higgs is dominated by  $H \rightarrow W_L^+W_L^-$  and  $H \rightarrow Z_LZ_L$ . You can base your considerations on the diagrams in parts (i) and (ii).

## 12.6 $H \rightarrow gg$

The Higgs can decay to a pair of gluons. The leading contribution comes from a top quark loop.

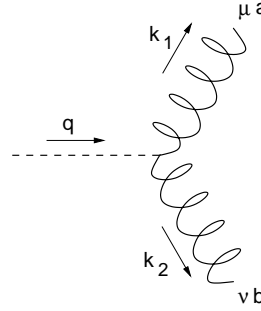


For  $m_H \ll m_t$  this process can be captured by a low-energy effective

Lagrangian with an interaction term

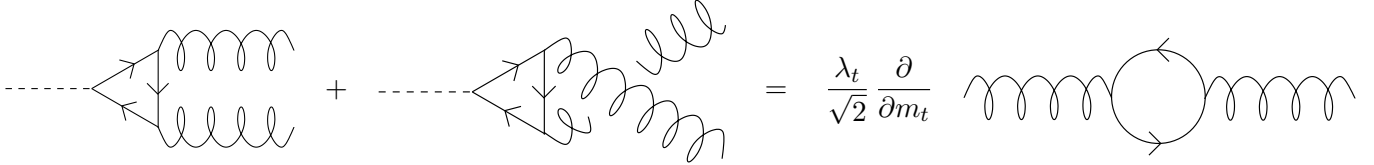
$$\mathcal{L}_{\text{int}} = \frac{1}{2} A H \text{Tr} (G_{\mu\nu} G^{\mu\nu}). \quad (12.4)$$

Here  $A$  is a coupling constant,  $H$  is the Higgs field, and  $G_{\mu\nu}$  is the gluon field strength. The corresponding interaction vertex is



$$-iA\delta^{ab}(k_1 \cdot k_2 g^{\mu\nu} - k_1^\nu k_2^\mu)$$

- (i) Write down the amplitude for the two triangle diagrams. No need to evaluate traces or loop integrals at this stage.
- (ii) Set  $q = 0$  so that  $k_1 = -k_2 \equiv k$  and show that



$$= \frac{\lambda_t}{\sqrt{2}} \frac{\partial}{\partial m_t} \text{ (vacuum polarization diagram) }$$

Here  $\lambda_t$  is the top quark Yukawa coupling and  $m_t$  is the top quark mass. Is there a simple reason you'd expect such a relation to hold?

- (iii) Use the results from appendix C to show that the vacuum polarization diagram is equal to

$$-\frac{2}{3}g^2\delta^{ab}(g^{\mu\nu}k^2 - k^\mu k^\nu) \int \frac{d^4p}{(2\pi)^4} \left[ \frac{1}{(p^2 - m_t^2)^2} - \frac{1}{(p^2 - M^2)^2} \right] + \mathcal{O}(k^4).$$

Here we're doing a Taylor series expansion in the external momentum  $k$ , and  $M$  is a Pauli-Villars regulator mass. (Alternatively you could work with a momentum cutoff  $\Lambda$  and send  $\Lambda \rightarrow \infty$ .) Use this to compute the amplitude for  $H \rightarrow gg$  at  $q = 0$ . Match to the amplitude you get from the effective field theory vertex and determine the coupling  $A$ .

- (iv) Use the effective Lagrangian to compute the partial width for the decay  $H \rightarrow gg$ . You should work on-shell, with  $q^2 = m_H^2$ .

Express your answer in terms of  $\alpha_s$ , the Higgs mass  $m_H$ , and the Higgs vev  $v$ .

A few comments: the effective Lagrangian (12.4) can also be used to describe the “gluon fusion” process  $gg \rightarrow H$  which is the main mechanism for producing the Higgs boson at a hadron collider. Note that the width we’ve obtained is independent of the top mass. In fact the calculation is valid in the limit  $m_t \rightarrow \infty$ . This violates the decoupling of heavy particles mentioned at the end of problem C.2. The reason is that large  $m_t$  indeed suppresses the loop, but large  $\lambda_t$  enhances the vertex, and these competing effects leave a finite result in the limit  $m_t \sim \lambda_t \rightarrow \infty$ .

# 13

## Anomalies

Physics 85200

January 6, 2014

Symmetries have played a crucial role in our construction of the standard model. So far by a symmetry we've meant a transformation of the fields that leaves the Lagrangian invariant. This was our definition of a symmetry in chapter 5. Although perfectly sensible in classical field theory, this definition misses a key aspect of the quantum theory, namely that quantum field theory requires both a Lagrangian and a cutoff procedure to be well-defined. It could be that symmetries of the Lagrangian are violated by the cutoff procedure. Sometimes such violations are inevitable, in which case the symmetry is said to be anomalous. The prototype for this sort of phenomenon is the “chiral anomaly:” the breakdown of gauge invariance in chiral spinor electrodynamics.

### 13.1 The chiral anomaly

Consider a free massless chiral fermion, either right- or left-handed. We will describe it using a Dirac spinor  $\psi$  with either the top two or bottom two components of  $\psi$  vanishing. The free Lagrangian  $\mathcal{L} = \bar{\psi}i\gamma^\mu\partial_\mu\psi$  has an obvious  $U(1)$  symmetry  $\psi \rightarrow e^{-i\alpha}\psi$ . The corresponding Noether current  $j^\mu = \bar{\psi}\gamma^\mu\psi$  is classically conserved; one can easily check that the Dirac equation  $\not{\partial}\psi = 0$  implies  $\partial_\mu j^\mu = 0$ . Following the standard procedure you might think we can gauge this symmetry, introducing a vector field  $A_\mu$  and a covariant derivative to obtain a theory

$$\mathcal{L} = \bar{\psi}i\gamma^\mu (\partial_\mu + ieQA_\mu) \psi \quad (13.1)$$

which is invariant under position-dependent gauge transformations

$$\psi \rightarrow e^{-ieQ\alpha(x)}\psi \quad A_\mu \rightarrow A_\mu + \partial_\mu\alpha.$$

As usual,  $Q$  is the charge of the field measured in units of  $e = \sqrt{4\pi\alpha}$ .

It might seem we've described one of the simplest gauge theories imaginable – massless chiral spinor electrodynamics. The classical Lagrangian (13.1) is certainly gauge invariant. But as we'll show, gauge invariance is spoiled by radiative corrections at the one loop level. This phenomenon is variously known as the chiral, triangle, or ABJ anomaly, after its discoverers Adler, Bell and Jackiw.<sup>†</sup>

Before discussing the breakdown of gauge invariance, it's important to realize that we're going to use the vector field in two different ways.

- (i) We might regard  $A_\mu$  as a classical background field. In this case the vector field has no dynamics of its own; rather it's value is prescribed externally to the system by some agent. We can then use  $A_\mu$  to probe the behavior of the system. For example, we can obtain the current  $j^\mu = \bar{\psi}\gamma^\mu\psi$  by varying the action with respect to the vector field.

$$j^\mu(x) = -\frac{1}{eQ} \frac{\delta S}{\delta A_\mu(x)} \quad (13.2)$$

- (ii) We might try to promote  $A_\mu$  to a dynamical field, adding a Maxwell term to the action and giving it a life (or at least, equations of motion) of its own.

Given the breakdown of gauge invariance, the second possibility cannot be realized.

### 13.1.1 Triangle diagram and shifts of integration variables

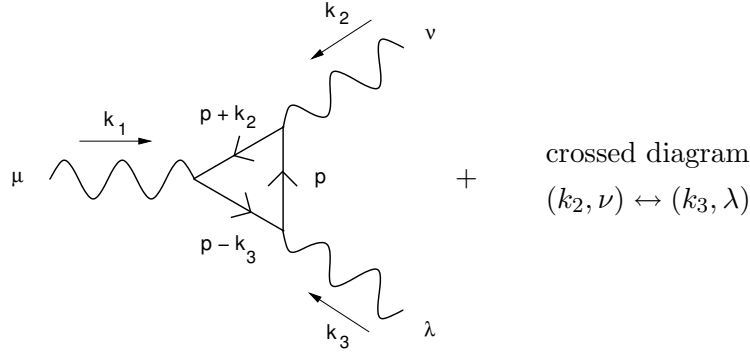
We now turn to the breakdown of gauge invariance. The problem with gauge invariance is rather subtle and unexpected (hence the name anomaly): it arises in, and only in, the one-loop triangle graph for three photon scattering. The Lagrangian (13.1) corresponds to a vertex



The diagram shows a triangle loop. Two fermion lines (solid lines with arrows) enter from the left and meet at a vertex. A photon line (wavy line) connects this vertex to another vertex on the right. A second fermion line enters from the bottom left and meets at a second vertex. The photon line also connects this second vertex to the first vertex, completing the triangle loop. To the right of the diagram is the expression  $-ieQ\gamma^\mu\frac{1}{2}(1 \pm \gamma^5)$ .

There's a projection operator in the vertex to enforce that only a single spinor chirality participates. *Throughout this chapter the upper sign corresponds to a right-handed spinor, the lower sign to left-handed.* This vertex leads to three photon scattering at one loop via the diagrams

<sup>†</sup> S. Adler, *Phys. Rev.* 177 (1969) 2426, J. Bell and R. Jackiw, *Nuovo Cim.* **60A** (1969) 47.



The scattering amplitude is easy to write down.

$$\begin{aligned}
 -i\mathcal{M}_{\mu\nu\lambda} = & (-1) \int \frac{d^4p}{(2\pi)^4} \text{Tr} \left\{ -ieQ\gamma_\mu \frac{i}{\not{p} + \not{k}_2} (-ieQ\gamma_\nu) \frac{i}{\not{p}} (-ieQ\gamma_\lambda) \frac{i}{\not{p} - \not{k}_3} \frac{1}{2}(1 \pm \gamma^5) \right\} \\
 & + \text{Tr} \left\{ -ieQ\gamma_\mu \frac{i}{\not{p} + \not{k}_3} (-ieQ\gamma_\lambda) \frac{i}{\not{p}} (-ieQ\gamma_\nu) \frac{i}{\not{p} - \not{k}_2} \frac{1}{2}(1 \pm \gamma^5) \right\} \quad (13.3)
 \end{aligned}$$

All external momenta are directed inward, with  $k_1 + k_2 + k_3 = 0$ . Also we combined the projection operators in each vertex into a single  $\frac{1}{2}(1 \pm \gamma^5)$  which enforces the fact that only a single spinor chirality circulates in the loop.

What properties do we expect of this amplitude?

- (i) Current conservation at each vertex, or equivalently gauge invariance. This implies that photons with polarization vectors proportional to their momentum should decouple,

$$k_1^\mu \mathcal{M}_{\mu\nu\lambda} = k_2^\nu \mathcal{M}_{\mu\nu\lambda} = k_3^\lambda \mathcal{M}_{\mu\nu\lambda} = 0.$$

- (ii) Bose statistics. Photons have spin 1, so the amplitude should be invariant under permutations of the external lines.

There's a simple argument which seems to show that Bose symmetry is satisfied. Invariance under exchange  $(k_2, \nu) \leftrightarrow (k_3, \lambda)$  is manifest; given our labelings it just corresponds to exchanging the two diagrams. However we should check invariance under exchange of say  $(k_1, \mu)$  with  $(k_2, \nu)$ . Making this exchange in (13.3) we get

$$\begin{aligned}
 -i\mathcal{M}_{\nu\mu\lambda} = & (-1) \int \frac{d^4p}{(2\pi)^4} \text{Tr} \left\{ -ieQ\gamma_\nu \frac{i}{\not{p} + \not{k}_1} (-ieQ\gamma_\mu) \frac{i}{\not{p}} (-ieQ\gamma_\lambda) \frac{i}{\not{p} - \not{k}_3} \frac{1}{2}(1 \pm \gamma^5) \right\} \\
 & + \text{Tr} \left\{ -ieQ\gamma_\nu \frac{i}{\not{p} + \not{k}_3} (-ieQ\gamma_\lambda) \frac{i}{\not{p}} (-ieQ\gamma_\mu) \frac{i}{\not{p} - \not{k}_1} \frac{1}{2}(1 \pm \gamma^5) \right\}
 \end{aligned}$$



Shifting the integration variable  $p^\mu \rightarrow p^\mu + k_3^\mu$  in the first line, and  $p^\mu \rightarrow p^\mu - k_3^\mu$  in the second, and making some cyclic permutations inside the trace, we seem to recover our original expression (13.3).

There's a similar argument for current conservation. Let's check whether  $k_1^\mu \mathcal{M}_{\mu\nu\lambda} = 0$ . Dotting the amplitude into  $k_1^\mu$  and using the trivial identities

$$\begin{aligned} \not{k}_1 &= \not{p} - \not{k}_3 - (\not{p} + \not{k}_2) && \text{in the first line} \\ \not{k}_1 &= \not{p} - \not{k}_2 - (\not{p} + \not{k}_3) && \text{in the second} \end{aligned} \quad (13.4)$$

we obtain

$$\begin{aligned} -ik_1^\mu \mathcal{M}_{\mu\nu\lambda} &= -e^3 Q^3 \int \frac{d^4 p}{(2\pi)^4} \text{Tr} \left\{ \frac{1}{\not{p} + \not{k}_2} \gamma_\nu \frac{1}{\not{p}} \gamma_\lambda \frac{1}{2} (1 \mp \gamma^5) - \frac{1}{\not{p} - \not{k}_3} \gamma_\nu \frac{1}{\not{p}} \gamma_\lambda \frac{1}{2} (1 \mp \gamma^5) \right. \\ &\quad \left. + \frac{1}{\not{p}} \gamma_\nu \frac{1}{\not{p} + \not{k}_3} \gamma_\lambda \frac{1}{2} (1 \mp \gamma^5) - \frac{1}{\not{p}} \gamma_\nu \frac{1}{\not{p} - \not{k}_2} \gamma_\lambda \frac{1}{2} (1 \mp \gamma^5) \right\} \end{aligned}$$

After shifting  $p \rightarrow p - k_2$  in the first term, it seems the first and fourth terms cancel. Likewise after shifting  $p \rightarrow p + k_3$  in the second term, it seems the second and third terms cancel.

This makes it seem we have both current conservation and Bose statistics. However our arguments relied on shifting the loop momentum and – this is the subtle point – one can't necessarily shift the integration variable in a divergent integral. To see this consider a generic loop integral  $\int \frac{d^4 p}{(2\pi)^4} f(p^\mu + a^\mu)$ . Suppose we expand the integrand in a Taylor series.

$$\int \frac{d^4 p}{(2\pi)^4} f(p^\mu + a^\mu) = \int \frac{d^4 p}{(2\pi)^4} f(p) + a^\mu \partial_\mu f(p) + \frac{1}{2} a^\mu a^\nu \partial_\mu \partial_\nu f(p) + \dots$$

If the integral converges, or is at most log divergent, then  $f(p)$  falls off rapidly enough at large  $p$  that we can drop total derivatives. This is the usual situation, and corresponds to the fact that usually the integral is independent of  $a^\mu$ . But if the integral diverges we need to have a cutoff in mind, say a cutoff on the magnitude of the Euclidean 4-momentum  $|p_E| < \Lambda$ . For linearly divergent integrals  $f(p) \sim 1/p^3$  and the order  $a$  term in the Taylor series generates a finite surface term. This invalidates the naive arguments for Bose symmetry and current conservation given above.

For future reference it's useful to be explicit about the value of the surface term. For a linearly divergent integral

$$\begin{aligned} \int \frac{d^4 p}{(2\pi)^4} a^\mu \partial_\mu f(p) &= -i \int_{|p_E| < \Lambda} \frac{d^4 p_E}{(2\pi)^4} a_E^\mu \frac{\partial}{\partial p_E^\mu} f(p_E) \\ &= -ia_E^\mu \int_0^\Lambda p_E^3 dp_E \int \frac{d\Omega}{(2\pi)^4} \frac{\partial}{\partial p_E^\mu} f(p_E) \end{aligned}$$

$$\begin{aligned}
&= -ia_E^\mu \int \frac{d\Omega}{(2\pi)^4} p_E^2 p_E^\mu f(p_E) \Big|_{p_E=\Lambda} \\
&= -ia_E^\mu \frac{1}{8\pi^2} \lim_{p_E \rightarrow \infty} \langle p_E^2 p_E^\mu f(p_E) \rangle \\
&= -ia_E^\mu \frac{1}{8\pi^2} \lim_{p \rightarrow \infty} \langle p^2 p_\mu f(p) \rangle \tag{13.5}
\end{aligned}$$

In the first line we Wick rotated to Euclidean space. In the second line we switched to spherical coordinates. In the third line we did the radial integral, picking up a unit outward normal vector  $p_E^\mu/p_E$ . In the fourth line we rewrote the angular integral as an average over a unit 3-sphere with “area”  $2\pi^2$  and took the limit  $\Lambda \rightarrow \infty$ . In the last line we rotated back to Minkowski space; the angle brackets now indicate an average over the Lorentz group.

### 13.1.2 Triangle diagram redux

Now that we’ve understood the potential difficulty, let’s return to the triangle diagrams. Rather than study the violation of Bose symmetry in detail, we’re simply going to demand that the scattering amplitude be symmetric. The most straightforward way to do this is to *define* the scattering amplitude to be given by averaging over all permutations of the external lines. Equivalently, we average over cyclic permutations of the internal momentum routing. That is, we define the Bose-symmetrized amplitude

$$\begin{aligned}
-i\mathcal{M}_{\mu\nu\lambda}^{\text{symm}} &= \frac{1}{3} \left[ \begin{array}{c} \text{Diagram 1: } \mu \text{ (wavy), } \nu \text{ (wavy), } \lambda \text{ (wavy), internal } p \\ \text{Diagram 2: } \mu \text{ (wavy), } \nu \text{ (wavy), } \lambda \text{ (wavy), internal } p \\ \text{Diagram 3: } \mu \text{ (wavy), } \nu \text{ (wavy), } \lambda \text{ (wavy), internal } p \end{array} \right. \\
&\quad \left. + \text{crossed diagrams} \right]
\end{aligned}$$

Explicitly this gives

$$\begin{aligned}
-i\mathcal{M}_{\mu\nu\lambda}^{\text{symm}} &= \mp \frac{1}{6} e^3 Q^3 \int \frac{d^4 p}{(2\pi)^4} \text{Tr} \left\{ \gamma_\mu \frac{1}{\not{p}} \gamma_\nu \frac{1}{\not{p} - \not{k}_2} \gamma_\lambda \frac{1}{\not{p} + \not{k}_1} \gamma^5 + \gamma_\mu \frac{1}{\not{p} + \not{k}_2} \gamma_\nu \frac{1}{\not{p}} \gamma_\lambda \frac{1}{\not{p} - \not{k}_3} \gamma^5 \right. \\
&\quad + \gamma_\mu \frac{1}{\not{p} - \not{k}_1} \gamma_\nu \frac{1}{\not{p} + \not{k}_3} \gamma_\lambda \frac{1}{\not{p}} \gamma^5 + \gamma_\mu \frac{1}{\not{p}} \gamma_\lambda \frac{1}{\not{p} - \not{k}_3} \gamma_\nu \frac{1}{\not{p} + \not{k}_1} \gamma^5 \\
&\quad \left. + \gamma_\mu \frac{1}{\not{p} + \not{k}_3} \gamma_\lambda \frac{1}{\not{p}} \gamma_\nu \frac{1}{\not{p} - \not{k}_2} \gamma^5 + \gamma_\mu \frac{1}{\not{p} - \not{k}_1} \gamma_\lambda \frac{1}{\not{p} + \not{k}_2} \gamma_\nu \frac{1}{\not{p}} \gamma^5 \right\}
\end{aligned}$$

Here we've used the fact that only terms involving  $\gamma^5$  contribute to the scattering amplitude.<sup>†</sup> The amplitude is divergent; to regulate it we'll impose a cutoff on the Euclidean loop momentum  $|p_E| < \Lambda$ .

Having enforced Bose symmetry, let's check current conservation by dotting this amplitude into  $k_1^\mu$ . Using identities similar to (13.4) to cancel the propagators adjacent to  $k_1$ , it turns out that most terms cancel, leaving only

$$\begin{aligned} -ik_1^\mu \mathcal{M}_{\mu\nu\lambda}^{\text{symm}} = & \pm \frac{1}{6} e^3 Q^3 \int \frac{d^4 p}{(2\pi)^4} \text{Tr} \left\{ \frac{1}{\not{p} + \not{k}_2} \gamma_\nu \frac{1}{\not{p} - \not{k}_1} \gamma_\lambda \gamma^5 - \frac{1}{\not{p} + \not{k}_1} \gamma_\nu \frac{1}{\not{p} - \not{k}_2} \gamma_\lambda \gamma^5 \right. \\ & \left. + \frac{1}{\not{p} - \not{k}_1} \gamma_\nu \frac{1}{\not{p} + \not{k}_3} \gamma_\lambda \gamma^5 - \frac{1}{\not{p} - \not{k}_3} \gamma_\nu \frac{1}{\not{p} + \not{k}_1} \gamma_\lambda \gamma^5 \right\} \end{aligned}$$

Shifting  $p \rightarrow p + k_2 - k_1$  in the second term it seems to cancel the first, and shifting  $p \rightarrow p + k_3 - k_1$  in the fourth term it seems to cancel the third. This naive cancellation means the whole expression is given just by a surface term.

$$\begin{aligned} -ik_1^\mu \mathcal{M}_{\mu\nu\lambda}^{\text{symm}} = & \pm \frac{1}{6} e^3 Q^3 \int \frac{d^4 p}{(2\pi)^4} (k_2^\alpha - k_1^\alpha) \frac{\partial}{\partial p^\alpha} \text{Tr} \left\{ \frac{1}{\not{p}} \gamma_\nu \frac{1}{\not{p} + \not{k}_3} \gamma_\lambda \gamma^5 \right\} \\ & + (k_3^\alpha - k_1^\alpha) \frac{\partial}{\partial p^\alpha} \text{Tr} \left\{ \frac{1}{\not{p} + \not{k}_2} \gamma_\nu \frac{1}{\not{p}} \gamma_\lambda \gamma^5 \right\} \end{aligned}$$

Using our result for the surface term (13.5), evaluating the Dirac traces with  $\text{Tr}(\gamma^\alpha \gamma^\beta \gamma^\gamma \gamma^\delta \gamma^5) = 4i\epsilon^{\alpha\beta\gamma\delta}$ , and averaging over the Lorentz group with  $\langle p_\alpha p_\beta \rangle = \frac{1}{4} g_{\alpha\beta} p^2$  we are left with a finite, non-zero, “anomalous” result.

$$-ik_1^\mu \mathcal{M}_{\mu\nu\lambda}^{\text{symm}} = \mp \frac{e^3 Q^3}{12\pi^2} \epsilon_{\nu\lambda\alpha\beta} k_2^\alpha k_3^\beta \quad (13.6)$$

Current conservation is violated by the triangle diagrams!

### 13.1.3 Comments

This breakdown of current conservation is quite remarkable, and there's quite a bit to say about it. Let me start by giving a few different ways to formulate the result.

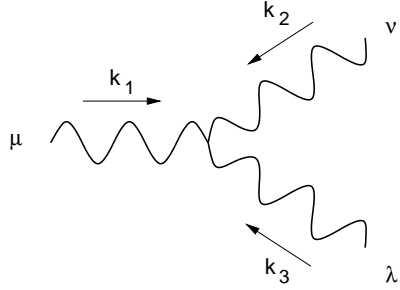
- (i) One could imagine writing down an effective action for the vector field  $\Gamma[A]$  which incorporates the effect of fermion loops. The amplitude

<sup>†</sup> Terms without a  $\gamma^5$  would describe three photon scattering in ordinary QED. But in ordinary QED the photon is odd under charge conjugation and the amplitude for three photon scattering vanishes (Furry's theorem).

we've computed corresponds to the following rather peculiar-looking term in the effective action.

$$\Gamma[A] = \cdots \pm \int d^4x d^4y \frac{e^3 Q^3}{96\pi^2} \partial_\mu A^\mu(x) \square^{-1}(x-y) \epsilon^{\alpha\beta\gamma\delta} F_{\alpha\beta} F_{\gamma\delta}(y) \quad (13.7)$$

Here  $\square^{-1}(x-y)$  should be thought of as a Green's function, the inverse of the operator  $\partial_\mu \partial^\mu$ , and  $F_{\alpha\beta}$  is the field strength of  $A_\mu$ . To verify this, note that the term we've written down in  $\Gamma[A]$  corresponds to a 3-photon vertex



$$\mp \frac{e^3 Q^3}{12\pi^2} \left( \frac{1}{k_1^2} k_{1\mu} \epsilon_{\nu\lambda\alpha\beta} k_2^\alpha k_3^\beta + \text{cyclic perms} \right)$$

When dotted into one of the external momenta, this amplitude reproduces (13.6).

- (ii) One can view the anomaly as a violation of current conservation. Without making  $A_\mu$  dynamical, we can regard it as an externally prescribed background field, and we can use it to define a quantum-corrected current via  $j^\mu = -\frac{1}{eQ} \frac{\delta\Gamma}{\delta A_\mu}$  (this parallels the classical current definition (13.2)). Given the term (13.7) in the effective action, the quantum-corrected current satisfies

$$\partial_\mu j^\mu = -\frac{1}{eQ} \partial_\mu \frac{\delta\Gamma}{\delta A_\mu} = \pm \frac{e^2 Q^2}{96\pi^2} \epsilon^{\alpha\beta\gamma\delta} F_{\alpha\beta} F_{\gamma\delta}. \quad (13.8)$$

- (iii) One can also view the anomaly as a breakdown of gauge invariance. Clearly the effective action (13.7) isn't gauge invariant. This means the gauge invariance of the classical Lagrangian is violated by radiative corrections. Since the gauge invariance is broken, it would not be consistent to promote  $A_\mu$  to a dynamical gauge field.

There's a connection between current conservation and gauge invariance: the divergence of the current measures the response of the effective action to a gauge transformation. To see this note that under  $A_\mu \rightarrow A_\mu + \partial_\mu \alpha$  we

have

$$\delta\Gamma = \int d^4x \frac{\delta\Gamma}{\delta A_\mu} \partial_\mu \alpha = - \int d^4x \partial_\mu \frac{\delta\Gamma}{\delta A_\mu} \alpha = eQ \int d^4x \partial_\mu j^\mu \alpha. \quad (13.9)$$

Next let me make a few comments on how robust our results are.

- (i) Due to the  $1/\square$ , the effective action we've written down is non-local (it can't be written with a single  $\int d^4x$ ). Although seemingly obscure, this is actually a very important point. Imagine modifying the behavior of our theory at short distances while keeping the long-distance behavior the same. To be concrete you could imagine that some new heavy particles, or even quantum gravity effects, become important at short distances. By definition such short-distance modifications can only affect *local* terms in the effective action. Since the term we wrote down is non-local, the anomaly is independent of any short-distance change in the dynamics!
- (ii) As we saw, the anomaly arises from the need to introduce an ultra-violet regulator, which can be thought of as an ad hoc short-distance modification to the dynamics. But given our statements above, the details of the regulator don't matter – any cutoff procedure will give the same result for the anomaly! (See however section 13.2.)
- (iii) The anomaly we've computed at one loop is not corrected by higher orders in perturbation theory. Our result for the divergence of the current (13.8) is exact! This is known as the Adler-Bardeen theorem.<sup>†</sup> The proof is based on showing that only the triangle diagram has the divergence structure necessary for generating an anomaly.

It's worth amplifying on the cutoff dependence. Field theory requires both an underlying Lagrangian and a cutoff scheme. A symmetry of the Lagrangian will be a symmetry of the effective action *provided the symmetry is respected by the cutoff*. Otherwise symmetry-breaking terms will be generated in the effective action. We saw an example of this in appendix C, where we used a momentum cutoff to compute the vacuum polarization diagram and found that an explicit photon mass term was generated. It could be that the symmetry breaking terms are local, as in appendix C, in which case they can be canceled by adding suitable “local counterterms” to the underlying action. Alternatively one could avoid generating the non-invariant terms in the first place by using a cutoff that respects the symmetry. But it could be that the symmetry-breaking terms in the effective action are non-local, as

<sup>†</sup> S. Adler and W. Bardeen, *Phys. Rev.* **182** (1969) 1517.

we found above for the anomaly. In this case no change in cutoff can restore the symmetry.

### 13.1.4 Generalizations

There are a few important generalizations of the triangle anomaly. First let's consider non-abelian symmetries. Take a chiral fermion, either right- or left-handed, in some representation of the symmetry group. Let  $T^a$  be a set of Hermitian generators. The label  $a$  could refer to global as well as to gauge symmetries. The current of interest is promoted to

$$j^{\mu a} = \bar{\psi} \gamma^\mu T^a \psi$$

and the vertex becomes



We're denoting the gauge coupling by  $g$ . The triangle graph can be evaluated just as in the abelian case and gives

$$\partial_\mu j^{\mu a} = \pm \frac{g^2}{96\pi^2} d^{abc} \epsilon^{\alpha\beta\gamma\delta} \left( \partial_\alpha A_\beta^b - \partial_\beta A_\alpha^b \right) \left( \partial_\gamma A_\delta^c - \partial_\delta A_\gamma^c \right) \quad \boxed{\text{triangle only}}$$

where  $d^{abc} = \frac{1}{2} \text{Tr} (T^a \{T^b, T^c\})$ . However for non-abelian symmetries the triangle graph isn't the end of the story: square and pentagon diagrams also contribute. If  $T^a$  generates a global symmetry, while  $T^b$  and  $T^c$  are gauge generators, then the full form of the anomalous divergence is easy to guess. We just promote the triangle result to the following gauge invariant form.

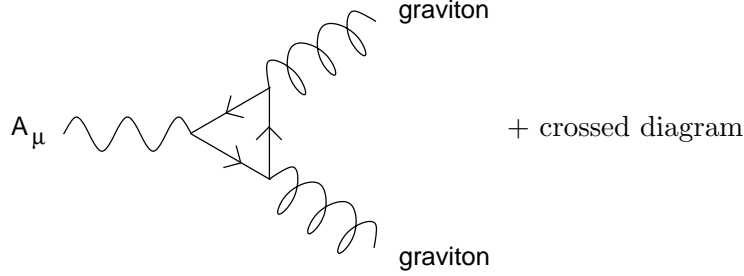
$$\partial_\mu j^{\mu a} = \pm \frac{g^2}{96\pi^2} d^{abc} \epsilon^{\alpha\beta\gamma\delta} F_{\alpha\beta}^b F_{\gamma\delta}^c \quad \boxed{\text{global}} \quad (13.10)$$

If  $T^a$  is one of the gauge generators then the full form of the anomaly is somewhat more involved. It turns out that the current is not covariantly conserved, but rather satisfies

$$\mathcal{D}_\mu j^{\mu a} = \pm \frac{g^2}{24\pi^2} d^{abc} \epsilon^{\alpha\beta\gamma\delta} \partial_\alpha \left( A_\beta^b \partial_\gamma A_\delta^c + \frac{1}{4} g f^{cde} A_\beta^b A_\gamma^d A_\delta^e \right) \quad \boxed{\text{gauge}}$$

where the structure constants of the group  $f^{abc}$  are defined by  $[T^a, T^b] = i f^{abc} T^c$ . In any case note that the anomaly is proportional to  $d^{abc}$ .

One can also get an anomaly from a triangle graph with one photon and two gravitons.



The photon couples to the current  $j^\mu = \bar{\psi}\gamma^\mu\psi$ . With a chiral fermion running in the loop, this diagram generates an anomalous divergence

$$\partial_\mu j^\mu = \pm \frac{1}{8 \cdot 96\pi^2} \epsilon^{\alpha\beta\gamma\delta} R_{\alpha\beta\sigma\tau} R_{\gamma\delta}{}^{\sigma\tau}$$

where  $R_{\alpha\beta\gamma\delta}$  is the Riemann curvature.<sup>†</sup>

### 13.2 Gauge anomalies

In order to gauge a symmetry we must have a valid global symmetry to begin with. To see how this might be achieved suppose we have two spinors, one right-handed and one left-handed. Assembling them into a Dirac spinor  $\psi$ , the currents

$$j_R^\mu = \bar{\psi}\gamma^\mu \frac{1}{2}(1 + \gamma^5)\psi \quad j_L^\mu = \bar{\psi}\gamma^\mu \frac{1}{2}(1 - \gamma^5)\psi$$

have anomalous divergences

$$\partial_\mu j_R^\mu = \frac{e^2 Q^2}{96\pi^2} \epsilon^{\alpha\beta\gamma\delta} R_{\alpha\beta} R_{\gamma\delta} \quad \partial_\mu j_L^\mu = -\frac{e^2 Q^2}{96\pi^2} \epsilon^{\alpha\beta\gamma\delta} L_{\alpha\beta} L_{\gamma\delta}. \quad (13.11)$$

Here  $R_\mu$  and  $L_\mu$  are background vector fields which couple to the chiral components of  $\psi$ , and quantities with two indices are the corresponding field strengths. Note that we've taken the right- and left-handed components of  $\psi$  to have the same charge. The vector and axial currents

$$j^\mu = j_R^\mu + j_L^\mu = \bar{\psi}\gamma^\mu\psi \quad j^{\mu 5} = j_R^\mu - j_L^\mu = \bar{\psi}\gamma^\mu\gamma^5\psi$$

<sup>†</sup> More precisely: in curved space the Dirac Lagrangian is  $\mathcal{L} = \bar{\psi}i\gamma^\mu(\partial_\mu + \frac{1}{2}\omega_\mu^{ab}\Sigma_{ab})\psi$  where  $\omega_\mu^{ab}$  is the spin connection and  $\Sigma_{ab} = \frac{1}{4}[\gamma_a, \gamma_b]$  are Lorentz generators. For a chiral fermion the  $A\omega\omega$  triangle graphs give  $\partial_\mu j^\mu = \pm \frac{1}{8 \cdot 96\pi^2} \epsilon^{\alpha\beta\gamma\delta} (\partial_\alpha \omega_\beta^{ab} - \partial_\beta \omega_\alpha^{ab})(\partial_\gamma \omega_{\delta ab} - \partial_\delta \omega_{\gamma ab})$  which can be promoted to the generally-covariant form  $\nabla_\mu j^\mu = \pm \frac{1}{8 \cdot 96\pi^2} \epsilon^{\alpha\beta\gamma\delta} R_{\alpha\beta\sigma\tau} R_{\gamma\delta}{}^{\sigma\tau}$ .

couple to the linear combinations

$$V_\mu = \frac{1}{2}(R_\mu + L_\mu) \quad A_\mu = \frac{1}{2}(R_\mu - L_\mu).$$

As a consequence of (13.11) these currents have divergences

$$\begin{aligned} \partial_\mu j^\mu &= \frac{e^2 Q^2}{24\pi^2} \epsilon^{\alpha\beta\gamma\delta} V_{\alpha\beta} A_{\gamma\delta} \\ \partial_\mu j^{\mu 5} &= \frac{e^2 Q^2}{48\pi^2} \epsilon^{\alpha\beta\gamma\delta} (V_{\alpha\beta} V_{\gamma\delta} + A_{\alpha\beta} A_{\gamma\delta}). \end{aligned}$$

At first sight this seems no better than having a single chiral spinor. But consider adding the following local term to the effective action for  $V_\mu$  and  $A_\mu$ .

$$\Delta S = \frac{ce^3 Q^3}{6\pi^2} \int d^4x \epsilon^{\alpha\beta\gamma\delta} \partial_\alpha V_\beta V_\gamma A_\delta$$

Here  $c$  is an arbitrary constant. This term violates both vector and axial gauge invariance, so it contributes to the divergences of the corresponding currents.

$$\begin{aligned} \Delta(\partial_\mu j^\mu) &= -\frac{1}{eQ} \partial_\mu \frac{\delta(\Delta S)}{\delta V_\mu} = -\frac{ce^2 Q^2}{24\pi^2} \epsilon^{\alpha\beta\gamma\delta} V_{\alpha\beta} A_{\gamma\delta} \\ \Delta(\partial_\mu j^{\mu 5}) &= -\frac{1}{eQ} \partial_\mu \frac{\delta(\Delta S)}{\delta A_\mu} = +\frac{ce^2 Q^2}{24\pi^2} \epsilon^{\alpha\beta\gamma\delta} V_{\alpha\beta} V_{\gamma\delta}. \end{aligned}$$

So if we add this term to the effective action and set  $c = 1$ , we have a conserved vector current but an anomalous axial current.

$$\partial_\mu j^\mu = 0 \quad \partial_\mu j^{\mu 5} = \frac{e^2 Q^2}{16\pi^2} \epsilon^{\alpha\beta\gamma\delta} \left( V_{\alpha\beta} V_{\gamma\delta} + \frac{1}{3} A_{\alpha\beta} A_{\gamma\delta} \right) \quad (13.12)$$

Given the conserved vector current we can add a Maxwell term to the action and promote  $V_\mu$  to a dynamical gauge field. The resulting theory is ordinary QED.<sup>†</sup>

QED is a simple example of gauge anomaly cancellation: the field content is adjusted so that the gauge anomalies cancel (that is, so that the effective action is gauge invariant). A similar cancellation takes place in any “vector-like” theory in which the right- and left-handed fermions have the same gauge quantum numbers. Anomaly cancellation in the standard model is more intricate because the standard model is a chiral theory: the left- and right-handed fermions have different gauge quantum numbers. In

<sup>†</sup> To make  $V_\mu$  dynamical the choice  $c = 1$  is mandatory and we have to live with the resulting anomalous divergence in the axial current. If we don’t make  $V_\mu$  dynamical then other choices for  $c$  are possible. This freedom corresponds to the freedom to use different cutoff procedures.



the standard model there are ten possible gauge anomalies, plus a gravitational anomaly, and we just have to check them all.

First let's consider the  $U(1)^3$  anomaly. A triangle diagram with three external  $U(1)_Y$  gauge bosons is proportional to  $\pm Y^3$ , where  $Y$  is the hypercharge of the fermion that circulates in the loop and the sign depends on whether the fermion is right- or left-handed. For the anomaly to cancel this must vanish when summed over all standard model fermions. For a single generation we have

$$\begin{aligned} \sum_{\text{left}} Y^3 &= \underbrace{(-1)^3}_{\nu_L} + \underbrace{(-1)^3}_{e_L} + \underbrace{3 \cdot (1/3)^3}_{u_L} + \underbrace{3 \cdot (1/3)^3}_{d_L} = -16/9 \\ \sum_{\text{right}} Y^3 &= \underbrace{(-2)^3}_{e_R} + \underbrace{3 \cdot (4/3)^3}_{u_R} + \underbrace{3 \cdot (-2/3)^3}_{d_R} = -16/9 \end{aligned}$$

The  $U(1)^3$  anomaly cancels! Note that three quark colors are required for this to work.

The full set of anomaly cancellation conditions are listed in the table. In general one has to show that the anomaly coefficient  $d^{abc}$  vanishes when appropriately summed over standard model fermions. In some cases the condition is rather trivial, since the  $SU(2)_L$  generators  $T_L^a = \frac{1}{2}\sigma^a$  and  $SU(3)_C$  generators  $T_C^a = \frac{1}{2}\lambda^a$  are both traceless (I'm being sloppy and using  $a$  to denote a generic group index). A few details: for the  $U(1)SU(2)^2$  anomaly right-handed fermions don't contribute, while  $\text{Tr } \sigma^a \sigma^b = 2\delta^{ab}$  is the same for every left-handed fermion, so we just get a condition on the sum of the left-handed hypercharges. Similarly for the  $U(1)SU(3)^2$  anomaly leptons don't contribute, while  $\text{Tr } \lambda^a \lambda^b = 2\delta^{ab}$  for every quark, so we just get a condition on the quark hypercharges.

Remarkably all conditions in the table are satisfied: the fermion content of the standard model is such that all potential gauge anomalies cancel. This cancellation provides some rational for the peculiar hypercharge assignments in the standard model. It's curious that both quarks and leptons are required for anomaly cancellation to work. However the anomalies cancel within each generation, so this provides no insight into Rabi's puzzle of who ordered the second generation.

### 13.3 Global anomalies

Gauge anomalies must cancel for a theory to be consistent. However anomalies in global symmetries are perfectly permissible, and indeed can have

anomaly	cancellation condition
$U(1)^3$	$\sum_{\text{left}} Y^3 = \sum_{\text{right}} Y^3$
$U(1)^2 SU(2)$	$\text{Tr } \sigma^a = 0$
$U(1)^2 SU(3)$	$\text{Tr } \lambda^a = 0$
$U(1) SU(2)^2$	$\sum_{\text{left}} Y = 0$
$U(1) SU(2) SU(3)$	$\text{Tr } \sigma^a = \text{Tr } \lambda^a = 0$
$U(1) SU(3)^2$	$\sum_{\text{left quarks}} Y = \sum_{\text{right quarks}} Y$
$SU(2)^3$	$\text{Tr } (\sigma^a \{ \sigma^b, \sigma^c \}) = \text{Tr } (\sigma^a) 2\delta^{bc} = 0$
$SU(2)^2 SU(3)$	$\text{Tr } \lambda^a = 0$
$SU(2) SU(3)^2$	$\text{Tr } \sigma^a = 0$
$SU(3)^3$	vector-like (left- and right-handed quarks in same representation)
$U(1) (\text{gravity})^2$	$\sum_{\text{left}} Y = \sum_{\text{right}} Y$

important physical consequences. To illustrate this I'll discuss global symmetries of the quark model. We'll encounter another example in section 14.2 when we discuss baryon and lepton number conservation.

Recall the quark model symmetries discussed in chapter 6. With two flavors of massless quarks  $\psi = \begin{pmatrix} u \\ d \end{pmatrix}$  we'd expect an  $SU(2)_L \times SU(2)_R$  chiral symmetry. Taking vector and axial combinations, the associated conserved currents are

$$j^{\mu a} = \bar{\psi} \gamma^\mu T^a \psi \quad j^{\mu 5a} = \bar{\psi} \gamma^\mu \gamma^5 T^a \psi \quad T^a = \frac{1}{2} \sigma^a.$$

To couple the quark model to electromagnetism we introduce the generator of  $U(1)_{\text{em}}$  which is just a matrix with quark charges along the diagonal.

$$Q = \begin{pmatrix} 2/3 & 0 \\ 0 & -1/3 \end{pmatrix}$$

However generalizing (13.12) to non-Abelian symmetries, along the lines of (13.10), we see that there is an  $SU(2)_A U(1)_{\text{em}}^2$  anomaly.

$$\partial_\mu j^{\mu 5a} = \frac{N_c e^2}{16\pi^2} \text{Tr} (T^a Q^2) \epsilon^{\alpha\beta\gamma\delta} F_{\alpha\beta} F_{\gamma\delta}$$

Here  $N_c = 3$  is the number of colors of quarks that run in the loop and  $F_{\alpha\beta}$  is the electromagnetic field strength. The anomaly is non-vanishing for the neutral pion ( $a = 3$ ). As you'll show on the homework, this is responsible for the decay  $\pi^0 \rightarrow \gamma\gamma$ .

The anomaly also lets us address a puzzle from chapter 6. With two flavors

of massless quarks the symmetry is really  $U(2)_L \times U(2)_R$ . The extra diagonal  $U(1)_V$  corresponds to conservation of baryon number. But what about the extra  $U(1)_A$ ? It's not a manifest symmetry of the particle spectrum, since the charge associated with  $U(1)_A$  would change the parity of any state it acted on and there are no even-parity scalars degenerate with the pions. Nor does  $U(1)_A$  seem to be spontaneously broken. The only obvious candidate for a Goldstone boson, the  $\eta$ , has a mass of 548 MeV and is too heavy to be regarded as a sort of "fourth pion."<sup>†</sup>

The following observation helps resolve the puzzle: the current associated with the  $U(1)_A$  symmetry,  $j^{\mu 5} = \bar{\psi} \gamma^\mu \gamma^5 \psi$ , has a triangle anomaly with two outgoing gluons.

$$\partial_\mu j^{\mu 5} = \frac{N_f g^2}{16\pi^2} \epsilon^{\alpha\beta\gamma\delta} \text{Tr} (G_{\alpha\beta} G_{\gamma\delta})$$

Here  $G_{\alpha\beta}$  is the gluon field strength and  $N_f = 2$  is the number of quark flavors. Since  $U(1)_A$  is not a symmetry of the quantum theory it would seem there is no need for a corresponding Goldstone boson. There are twists and turns in trying to make this argument precise, but in a weak-coupling expansion 't Hooft showed that the anomaly combined with topologically non-trivial gauge fields eliminates the need for a Goldstone boson associated with  $U(1)_A$ .

## References

Many classic papers on anomalies are reprinted in S.B. Treiman, R. Jackiw, B. Zumino and E. Witten, *Current algebra and anomalies* (Princeton, 1985). A textbook has been devoted to the subject: R. Bertlmann, *Anomalies in quantum field theory* (Oxford, 1996). The anomaly was discovered in studies of the decay  $\pi^0 \rightarrow \gamma\gamma$  by S. Adler, *Phys. Rev.* **177** (1969) 2426 and by J. Bell and R. Jackiw, *Nuovo Cim.* **60A** (1969) 47. The fact that the anomaly receives no radiative corrections was established by S. Adler and W. Bardeen, *Phys. Rev.* **182** (1969) 1517. The non-abelian anomaly was evaluated by W. Bardeen, *Phys. Rev.* **184** (1969) 1848. Gravitational anomalies were studied by L. Alvarez-Gaumé and E. Witten, *Nucl. Phys.* **B234** (1983) 269. A thorough discussion of anomaly cancellation can be found in Weinberg section 22.4. The role of the anomaly in resolving the  $U(1)_A$  puzzle was emphasized by G. 't Hooft, *Phys. Rev. Lett.* **37** (1976) 8. Topologically

<sup>†</sup> This objection can be made precise, see Weinberg section 19.10. A similar puzzle arises with three flavors of light quarks, where the  $\eta'$  with a mass of 958 MeV is too heavy to be grouped with members of the pseudoscalar meson octet.

non-trivial gauge fields play a crucial role in 't Hooft's analysis, as reviewed by S. Coleman, *Aspects of Symmetry* (Cambridge, 1985) chapter 7. A sober assessment of the status of the  $U(1)_A$  puzzle can be found in Donoghue section VII-4.

**WESS-ZUMINO TERMS.** The anomalous interactions of Goldstone bosons are described by so-called Wess-Zumino terms in the effective action. The structure of these terms was elucidated by E. Witten, *Nucl. Phys.* **B223** (1983) 422; for a path integral derivation see Donoghue section VII-3. For applications to  $\pi^0 \rightarrow \gamma\gamma$  see problem 13.1 and Donoghue section VI-5.

**REAL REPRESENTATIONS AND SAFE GROUPS.** Consider a group  $G$  and a representation  $\mathcal{D}(g) = e^{i\lambda^a T^a}$ ,  $g \in G$ . To have a unitary representation,  $\mathcal{D}\mathcal{D}^\dagger = \mathbf{1}$ , the generators  $T^a$  must be Hermitean. If we further require that the representation be real,  $\mathcal{D} = \mathcal{D}^*$ , then the generators  $T^a$  must be imaginary and antisymmetric. For a real representation

$$\mathrm{Tr} T^a T^b T^c = \mathrm{Tr} (T^a T^b T^c)^T = -\mathrm{Tr} T^c T^b T^a = -\mathrm{Tr} T^a T^c T^b$$

and the anomaly coefficient  $d^{abc} = \frac{1}{2} \mathrm{Tr} T^a \{T^b, T^c\}$  vanishes. The same conclusion holds if  $T^a = -U^\dagger (T^a)^T U$  for some unitary matrix  $U$ ; this covers both real representations (where one can take  $U = \mathbf{1}$ ) and pseudo-real representations (where one can't). All  $SU(2)$  representations are either real or pseudo-real, since one can map a representation to its complex conjugate using two-index  $\epsilon$  tensors. So  $SU(2)$  is an example of a “safe” group: the  $SU(2)^3$  anomaly vanishes in any representation. For a list of safe groups see H. Georgi and S. L. Glashow, *Phys. Rev.* **D6** (1972) 429 and F. Gursey, P. Ramond and P. Sikivie, *Phys. Lett.* **B60** (1976) 177.

## Exercises

### 13.1 $\pi^0 \rightarrow \gamma\gamma$

Recall that at low energies pions are described by the effective Lagrangian

$$\mathcal{L} = \frac{1}{4} f_\pi^2 \mathrm{Tr} \left( \partial_\mu U^\dagger \partial^\mu U \right)$$

where  $f_\pi = 93 \text{ MeV}$  and  $U = e^{i\vec{\pi} \cdot \vec{\sigma} / f_\pi}$  is an  $SU(2)$  matrix. This action has an  $SU(2)_L \times SU(2)_R$  symmetry  $U \rightarrow LUR^\dagger$ .

(i) Consider an infinitesimal  $SU(2)_A$  transformation for which

$$L \approx 1 - i\theta^a \sigma^a / 2 \quad R \approx 1 + i\theta^a \sigma^a / 2.$$

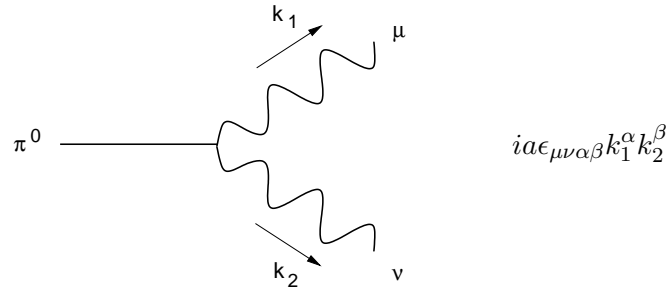
How do the pion fields  $\pi^a$  behave under this transformation? How does the effective action  $\Gamma$  behave under this transformation? Hint: plug the divergence of the axial current (13.12) into (13.9). You only need to keep track of terms involving two photons.

- (ii) Show that the anomaly can be taken into account by adding the following term to the effective Lagrangian.

$$\Delta\mathcal{L} = \frac{1}{8}a\pi^3\epsilon^{\alpha\beta\gamma\delta}F_{\alpha\beta}F_{\gamma\delta} \quad (13.13)$$

Determine the constant  $a$  by matching to your results in part (i).

- (iii) The anomalous term in the effective action corresponds to a vertex



Use this to compute the width for the decay  $\pi^0 \rightarrow \gamma\gamma$ . How did you do compared to the observed width  $7.7 \pm 0.5 \text{ eV}$ ?

A few comments on the calculation:

- Note that the  $\pi^0$  width is proportional to the number of quark colors.
- The anomaly dominates  $\pi^0$  decay because it induces a direct (non-derivative) coupling between the pion field and two Maxwell field strengths. Without the anomaly the pion would be a genuine Goldstone boson, with a shift symmetry  $\pi^a \rightarrow \pi^a + \theta^a$  that only allows for derivative couplings. The decay would then proceed via a term in the effective action of the form  $\partial_\mu \partial^\mu \pi^3 \epsilon^{\alpha\beta\gamma\delta} F_{\alpha\beta} F_{\gamma\delta}$ . As discussed by Weinberg p. 361 this would suppress the decay rate by an additional factor  $\sim (m_\pi/4\pi f_\pi)^4$ , where  $4\pi f_\pi \sim 1 \text{ GeV}$  is the scale associated with chiral perturbation theory introduced on p. 80.
- As discussed in the references, the term (13.13) in the effective action can be extended to a so-called Wess-Zumino term which fully incorporates the effects of the anomaly in the low energy dynamics of Goldstone bosons.

13.2 **Anomalous  $U(1)$ 's**

In QED the gauge anomaly cancels between the left- and right-handed components of the electron. There's another way to cancel anomalies in  $U(1)$  gauge theories, discovered by Green and Schwarz in the context of string theory. Consider an abelian gauge field  $A_\mu$  coupled to a chiral fermion, either left- or right-handed, so that the effective action has the anomalous gauge variation (13.8) and (13.9). Introduce a scalar field  $\phi$  which shifts under gauge transformations:

$$\phi \rightarrow \phi - \kappa\alpha \quad \text{when} \quad A_\mu \rightarrow A_\mu + \partial_\mu\alpha.$$

Here  $\kappa$  is a parameter with units of mass.

(i) Add the following higher-dimension term to the action.

$$\Delta S = \pm \frac{e^3 Q^3}{96\pi^2 \kappa} \phi \epsilon^{\alpha\beta\gamma\delta} F_{\alpha\beta} F_{\gamma\delta}$$

Show that under a gauge transformation the variation of  $\Delta S$  exactly cancels the anomalous variation of the effective action due to the triangle graph.

(ii) Potential terms for  $\phi$  are ruled out by the shift symmetry. Add a kinetic term for  $\phi$  to the action,  $\frac{1}{2}\mathcal{D}_\mu\phi\mathcal{D}^\mu\phi$  where  $\mathcal{D}_\mu$  is a suitable covariant derivative. Go to unitary gauge by setting  $\phi = 0$  and determine the mass spectrum of the theory.

# 14

## Additional topics

Physics 85200

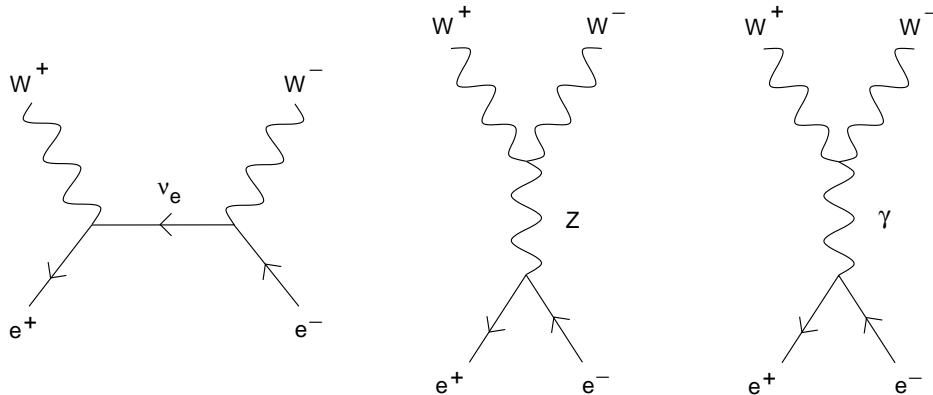
January 6, 2014

There are a few more features of the standard model I'd like to touch on before concluding. Each of these topics could be developed in much more detail. Some aspects are discussed in the homework; for further reading see the references.

### 14.1 High energy behavior

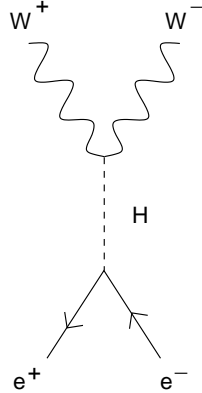
In section 9.4 we showed that the IVB theory of weak interactions suffers from bad high-energy behavior: although the IVB cross section for inverse muon decay is acceptable, the cross section for  $e^+e^- \rightarrow W^+W^-$  is in conflict with unitarity. We went on to construct the standard model as a spontaneously broken gauge theory, claiming that this would guarantee good high energy behavior. Here I'll give some evidence to support this claim. Rather than give a general proof of unitarity, I'll proceed by way of two examples.

Our first example is  $e^+e^- \rightarrow W^+W^-$ . In the standard model, provided one neglects the electron mass, there are three diagrams that contribute.



The first diagram, involving neutrino exchange, was studied in section 9.4 in the context of IVB theory. The other two diagrams, involving photon and  $Z$  exchange, are new features of the standard model. Each of these diagrams individually gives an amplitude that grows linearly with  $s$  at high energy. But the leading behavior cancels when the diagrams are added: the sum is independent of  $s$  in the high-energy limit, as required by unitarity. While theoretically satisfying, this cross section has also been measured at LEP. As can be seen in Fig. 14.1 the predictions of the standard model are borne out. This measurement can be regarded as a direct test of the  $ZWW$  and  $\gamma WW$  couplings. It shows that the weak interactions really are described by a non-abelian gauge theory!

The story becomes theoretically more interesting if we keep track of the electron mass. Then the diagrams above have subleading behavior  $\sim m_e s^{1/2}$  which does not cancel in the sum. Fortunately in the standard model there is an additional diagram involving Higgs exchange which contributes precisely when  $m_e \neq 0$ .



This diagram precisely cancels the  $s^{1/2}$  growth of the amplitude. The Higgs particle is necessary for unitarity! Unfortunately the electron mass is so small that we can't see the contribution of this diagram at LEP energies.

Another process, more interesting from a theoretical point of view but less accessible to experiment, is scattering of longitudinally-polarized  $W$  bosons,  $W_L^+ W_L^- \rightarrow W_L^+ W_L^-$ . In the standard model the tree-level amplitude for this process has the high-energy behavior<sup>†</sup>

$$\mathcal{M} = \frac{m_H^2}{v^2} \left( \frac{s}{s - m_H^2} + \frac{t}{t - m_H^2} \right). \quad (14.1)$$

<sup>†</sup> M. Duncan, G. Kane and W. Repko, *Nucl. Phys.* **B272** (1986) 517.



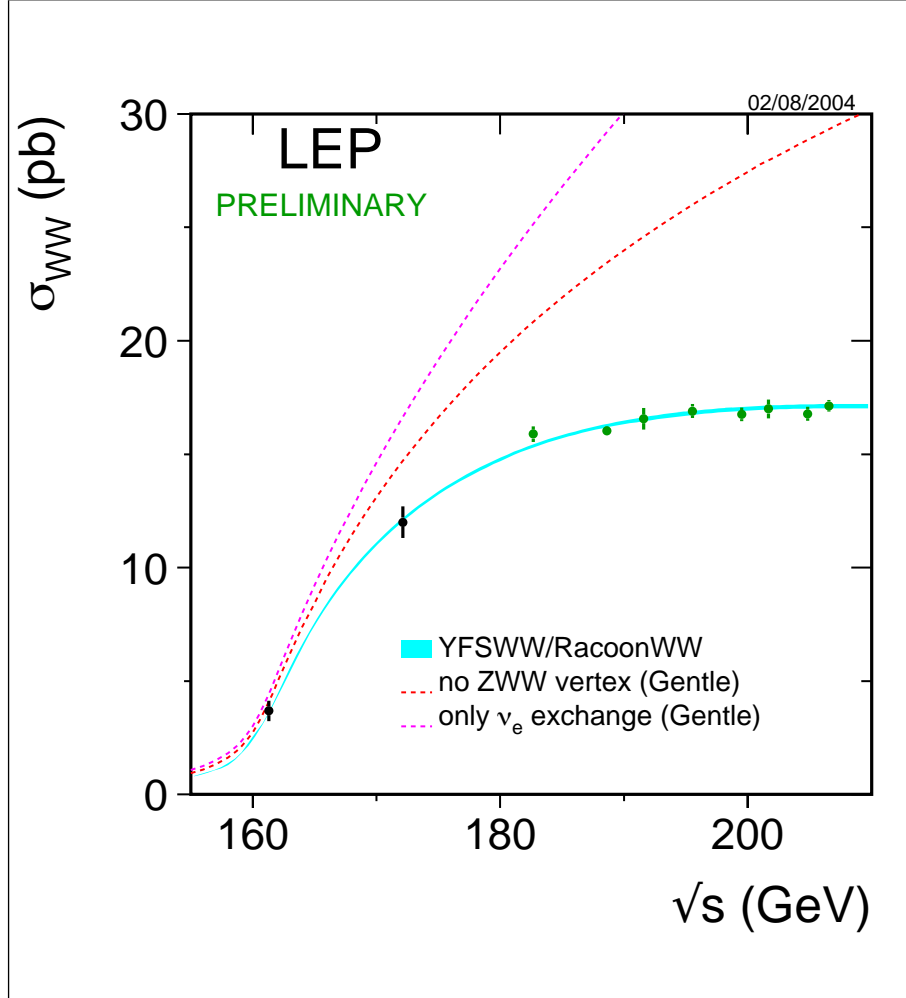


Fig. 14.1. The cross section for  $e^+e^- \rightarrow W^+W^-$  as measured at LEP. Data points and error bars are indicated. The solid blue curve is the standard model prediction. The dotted curves show what happens if the contributions of the  $\gamma$  and  $Z$  bosons are neglected. From the LEP electroweak working group, via C. Quigg arXiv:hep-ph/0502252.

To see the consequences of this result, it's useful to think about it in two different ways.

First way: suppose we require that the tree-level result (14.1) be compatible with unitarity at arbitrarily high energies. To study this we send  $s, t \rightarrow \infty$  and find  $\mathcal{M} \approx 2m_H^2/v^2$ . The unitarity bound on an  $s$ -wave cross-section  $\sigma_0 \leq 4\pi/s$  translates into a bound on the corresponding amplitude,

$|\mathcal{M}_0| \leq 8\pi$ . Imposing this requirement gives an upper bound on the Higgs mass,

$$m_H \leq \sqrt{4\pi}v = 870 \text{ GeV} . \quad (14.2)$$

If the Higgs mass satisfies this bound the standard model could in principle be extrapolated to arbitrarily high energies while remaining weakly coupled. Of course this may not be a sensible requirement to impose; if nothing else gravity should kick in at the Planck scale.

Second way: let's discard the physical Higgs particle by sending  $m_H \rightarrow \infty$ , and ask if anything goes wrong with the standard model. At large Higgs mass the amplitude (14.1) becomes

$$\mathcal{M} \approx -\frac{1}{v^2}(s+t) = -\frac{s}{v^2} (1 - \sin^2(\theta/2)) .$$

The  $s$ -wave amplitude is given by averaging this over scattering angles,

$$\mathcal{M}_0 = \frac{1}{4\pi} \int d\Omega \mathcal{M} = -\frac{s}{2v^2} .$$

The unitarity bound  $|\mathcal{M}_0| \leq 8\pi$  then implies  $\sqrt{s} \leq \sqrt{16\pi}v = 1.7 \text{ TeV}$ . That is, throwing out the standard model Higgs particle means that tree-level unitarity is violated at the TeV scale. *Something* must kick in before this energy scale in order to make  $W - W$  scattering compatible with unitarity. This is good news for the LHC: at the TeV scale either the standard model Higgs will be found, or some other new particles will be discovered, or at the very least strong-coupling effects will set in. However we should keep in mind that the LHC can't directly study  $W - W$  scattering, and unitarity bounds in other channels are weaker.

## 14.2 Baryon and lepton number conservation

We constructed the standard model by postulating a set of fields and writing down the most general gauge-invariant Lagrangian. However we only considered operators with mass dimension up to 4. One might argue that this is necessary for renormalizability, however there's no real reason to insist that the standard model be renormalizable. A better argument for stopping at dimension 4 is that any higher dimension operators we might add will have a negligible effect at low energies, provided they're suppressed by a sufficiently large mass scale.

Stopping with dimension-4 operators does have a remarkable consequence:

the standard model Lagrangian is invariant under  $U(1)$  symmetries corresponding to conservation of baryon and lepton number, as well as to conservation of the individual lepton flavors  $L_e, L_\mu, L_\tau$ . Note that we never had to postulate any of these conservation laws, rather they arise as a by-product of the field content of the standard model and the fact that we stopped at dimension 4. Such symmetries, which arise only because one restricts to renormalizable theories, are known as “accidental symmetries.”

These accidental symmetries of the standard model are phenomenologically desirable, of course, but there’s no reason to think they’re fundamental. There are two aspects to this.

- (i) It’s natural to imagine adding higher-dimension operators to the standard model, perhaps to reflect the effects of some underlying short-distance physics. There’s no reason to expect these higher-dimension operators to respect conservation of baryon or lepton number.
- (ii) The accidental symmetries of the dimension-4 Lagrangian lead to *classical* conservation of baryon and lepton number. However there’s no reason to expect that these conservation laws are respected by the quantum theory – there could be an anomaly.

We’ll see an explicit example of lepton number violation by higher dimension operators when we discuss neutrino masses in the next section. So let me focus on the second possibility, and show that the baryon and lepton number currents in the standard model indeed have anomalies.

To set up the problem, recall that the baryon number current  $j_B^\mu$  is one third of the quark number current. It can be written as a sum of left- and right-handed pieces.

$$j_B^\mu = \frac{1}{3} \sum_i (\bar{Q}_i \gamma^\mu Q_i + \bar{u}_{Ri} \gamma^\mu u_{Ri} + \bar{d}_{Ri} \gamma^\mu d_{Ri})$$

Here  $Q_i$  contains the left-handed quarks and  $i = 1, 2, 3$  is a generation index. There are also individual lepton flavor numbers, as well as the total lepton number, corresponding to currents

$$j_{Li}^\mu = \bar{L}_i \gamma^\mu L_i + \bar{e}_{Ri} \gamma^\mu e_{Ri} \qquad j_L^\mu = \sum_i j_{Li}^\mu$$

These currents have anomalies with electroweak gauge bosons. Making use

of the anomaly (13.10), the baryon number current has a divergence

$$\partial_\mu j_B^\mu = \frac{1}{3} N_c N_g \left[ \left( \sum_{\text{right}} Y^2 - \sum_{\text{left}} Y^2 \right) \frac{(g'/2)^2}{96\pi^2} \epsilon^{\alpha\beta\gamma\delta} B_{\alpha\beta} B_{\gamma\delta} - \frac{g^2}{96\pi^2} \epsilon^{\alpha\beta\gamma\delta} \text{Tr} (W_{\alpha\beta} W_{\gamma\delta}) \right] \quad (14.3)$$

where  $N_c = 3$  is the number of colors,  $N_g = 3$  is the number of generations, and the hypercharges of a single generation of quarks contribute a factor

$$\sum_{\text{right}} Y^2 - \sum_{\text{left}} Y^2 = \underbrace{(4/3)^2}_{u_R} + \underbrace{(-2/3)^2}_{d_R} - \underbrace{(1/3)^2}_{u_L} - \underbrace{(1/3)^2}_{d_L} = 2.$$

Likewise the individual lepton number currents have anomalous divergences

$$\partial_\mu j_{L_i}^\mu = \left( \sum_{\text{right}} Y^2 - \sum_{\text{left}} Y^2 \right) \frac{(g'/2)^2}{96\pi^2} \epsilon^{\alpha\beta\gamma\delta} B_{\alpha\beta} B_{\gamma\delta} - \frac{g^2}{96\pi^2} \epsilon^{\alpha\beta\gamma\delta} \text{Tr} (W_{\alpha\beta} W_{\gamma\delta}) \quad (14.4)$$

where a single generation of leptons gives

$$\sum_{\text{right}} Y^2 - \sum_{\text{left}} Y^2 = \underbrace{(-2)^2}_{e_R} - \underbrace{(-1)^2}_{\nu_L} - \underbrace{(-1)^2}_{e_L} = 2.$$

Curiously the right hand side of (14.4) is the same as the right hand side of (14.3), aside from an overall factor of  $\frac{1}{3} N_c N_g$ .

This shows that baryon number, as well as the individual lepton numbers, are all violated in the standard model. So why don't we observe baryon and lepton number violation? For simplicity let's focus on baryon number violation by hypercharge gauge fields. Given the anomalous divergence (14.3) we can find the change in baryon number between initial and final times  $t_i$ ,  $t_f$  by integrating

$$\Delta B = \int_{t_i}^{t_f} dt \int d^3x \partial_\mu j_B^\mu \sim \int_{t_i}^{t_f} dt \int d^3x \epsilon^{\alpha\beta\gamma\delta} B_{\alpha\beta} B_{\gamma\delta}.$$

But noting the identity

$$\epsilon^{\alpha\beta\gamma\delta} B_{\alpha\beta} B_{\gamma\delta} = \partial_\alpha \left[ 4\epsilon^{\alpha\beta\gamma\delta} B_\beta \partial_\gamma B_\delta \right]$$

we see that the change in baryon number is the integral of a total derivative. It's tempting to discard surface terms and conclude that baryon number is conserved. A more careful analysis shows that baryon number really is violated, but only in topologically non-trivial field configurations where the fields do not fall off rapidly enough at infinity to justify discarding surface terms. 't Hooft studied the resulting baryon number violation and showed

that at low energies it occurs at an unobservably small rate.<sup>†</sup> It's worth noting that the differences  $L_i - L_j$  are exactly conserved in the standard model, as is the combination  $B - L$ .<sup>‡</sup>

### 14.3 Neutrino masses

Another accidental feature of the (renormalizable, dimension-4) standard model is that neutrinos are massless. This is due to the field content of the standard model, in particular the fact that we never introduced right-handed neutrinos. However the observed neutrino flavor oscillations seem to require non-zero neutrino masses at the sub-eV level. To accommodate neutrino masses one approach is to extend the standard model by introducing a set of right-handed neutrinos  $\nu_{Ri}$  which we take to be singlets under the standard model gauge group (and therefore very hard to detect). We could then add a term to the standard model Lagrangian

$$\mathcal{L}_{\nu \text{ mass}} = -\Lambda_{ij}^\nu \bar{L}_i \tilde{\phi} \nu_{Rj} + \text{c.c.}$$

After electroweak symmetry breaking this would give the neutrinos a conventional Dirac mass term, via the same mechanism used for the up-type quarks. In this approach the small neutrino masses would be due to tiny Yukawa couplings  $\Lambda_{ij}^\nu$ . But extending the standard model in this way is somewhat awkward: we have no evidence for right-handed neutrinos, and it's hard to see why their Yukawa couplings should be so small.

There's a more appealing approach, in which we stick with the usual standard model field content but consider the effects of higher-dimension operators. The leading effects should come from dimension-5 operators. Remarkably, there's a unique operator one can write down at dimension 5 that's gauge invariant and built from the usual standard model fields. To construct the operator, note that  $\tilde{\phi}$  and  $L_i$  have exactly the same gauge quantum numbers. Then  $\tilde{\phi}^\dagger L_i$  is a left-handed spinor that is invariant under gauge transformations. Charge conjugation on a Dirac spinor acts by  $\psi_C = -i\gamma^2 \psi^*$  and has the effect of changing spinor chirality (see appendix D). This lets us build a gauge-invariant right-handed spinor

$$\left( \tilde{\phi}^\dagger L_i \right)_C = \tilde{\phi}^T L_{iC} .$$

<sup>†</sup> G. 't Hooft, *Phys. Rev. Lett.* **37** (1976) 8.

<sup>‡</sup> Strictly speaking the  $B - L$  current has a gravitational anomaly which could be canceled by adding right-handed neutrinos to the standard model.

Putting these spinors together, we can make a gauge-invariant bilinear<sup>†</sup>

$$\left(\tilde{\phi}^T L_{iC}\right) \left(\tilde{\phi}^\dagger L_j\right) = \left(\overline{L_{iC}} \tilde{\phi}^*\right) \left(\tilde{\phi}^\dagger L_j\right) .$$

So a possible dimension-5 addition to the standard model Lagrangian – in fact the only dimension-5 term allowed by gauge symmetry – is

$$\mathcal{L}_{\text{dim. 5}} = -\frac{\gamma_{ij}}{X} \left(\overline{L_{iC}} \tilde{\phi}^*\right) \left(\tilde{\phi}^\dagger L_j\right) - \frac{\gamma_{ij}^*}{X} \left(\overline{L_i} \tilde{\phi}\right) \left(\tilde{\phi}^T L_{jC}\right) . \quad (14.5)$$

Here  $\gamma_{ij}$  is a matrix of dimensionless coupling constants,  $X$  is a quantity with units of mass, and we added the complex conjugate to keep the Lagrangian real.<sup>†</sup>

After electroweak symmetry breaking we can plug in the Higgs vev  $\langle\tilde{\phi}\rangle = \begin{pmatrix} v/\sqrt{2} \\ 0 \end{pmatrix}$  and the lepton doublet  $L_i = \begin{pmatrix} \nu_{Li} \\ e_{Li} \end{pmatrix}$  to find that  $\mathcal{L}_{\text{dim. 5}}$  reduces to

$$-\frac{v^2}{2X} \gamma_{ij} \overline{\nu_{LiC}} \nu_{Lj} - \frac{v^2}{2X} \gamma_{ij}^* \overline{\nu_{Li}} \nu_{LjC} .$$

This is a so-called Majorana mass term for the neutrinos. It can be written more cleanly using the two-component notation introduced in appendix D, as

$$\frac{v^2}{2X} \gamma_{ij} \nu_i^T \epsilon \nu_j - \frac{v^2}{2X} \gamma_{ij}^* \nu_i^\dagger \epsilon \nu_j^* \quad (14.6)$$

where the Dirac spinor  $\nu_{Li} \equiv \begin{pmatrix} \nu_i \\ 0 \end{pmatrix}$ . In any case we can read off the neutrino mass matrix

$$m_{ij}^\nu = \frac{v^2}{X} \gamma_{ij} .$$

Assuming the energy scale  $X$  is much larger than the Higgs vev  $v$ , small Majorana neutrino masses  $m_\nu \sim v^2/X$  are to be expected in the standard model.

A few comments:

- (i) The dimension-5 operator we wrote down violates lepton number by two units, and generically also violates conservation of the individual lepton flavors  $L_e$ ,  $L_\mu$ ,  $L_\tau$ . This illustrates the fact that these quantities were only conserved due to accidental symmetries of the renormalizable standard model.

<sup>†</sup> The notation is a little overburdened: for example  $\overline{L_{iC}} \equiv (L_{iC})^\dagger \gamma^0 = (-i\gamma^2 L_i^*)^\dagger \gamma^0$ .

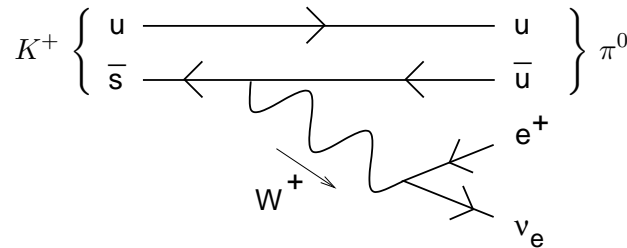
<sup>†</sup>  $\overline{L_{iC}} L_j$  is symmetric on  $i$  and  $j$  due to Fermi statistics, so we can take  $\gamma_{ij}$  to be symmetric as well.

- (ii) Once the neutrinos acquire a mass their gauge eigenstates and mass eigenstates can be different. This provides a mechanism for the observed phenomenon of neutrino flavor oscillations. (By gauge eigenstates I mean the states  $\nu_e, \nu_\mu, \nu_\tau$  that form  $SU(2)_L$  doublets with the charged leptons.)

### 14.4 Quark flavor violation

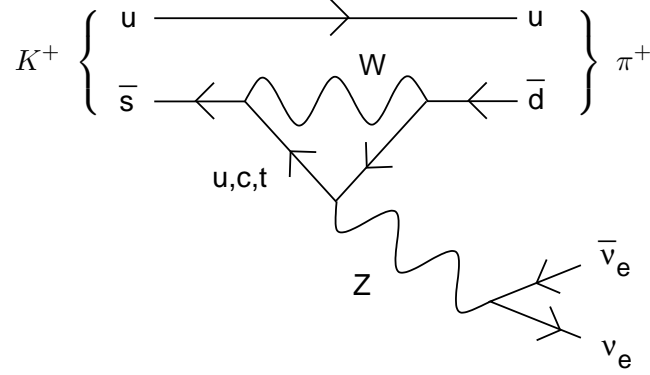
As we've seen, lepton flavor is accidentally conserved in the standard model. Quark flavor, on the other hand, is violated at the renormalizable level. But the flavor violation has a rather restricted form: as we saw in section 12.3 it only occurs via the CKM matrix in the quark – quark –  $W^\pm$  couplings. A surprising feature of the standard model is that the couplings of the  $Z$  conserve flavor – that is, that there are no flavor-changing neutral currents in the standard model.

The absence of flavor-changing neutral currents means that certain flavor-violating processes, while allowed, can only occur at the loop level. What's more, the loop diagrams often turn out to be anomalously small, due to an approximate cancellation known as the “GIM mechanism.” This can be nicely illustrated with kaon decays. First consider the decay  $K^+ \rightarrow \pi^0 e^+ \nu_e$ , which has an observed branching ratio of around 5%. At the quark level this decay occurs via a tree diagram involving  $W$  exchange.



Compare this to the decay  $K^+ \rightarrow \pi^+ \bar{\nu}_e \nu_e$ . Due to the absence of flavor-

changing neutral currents, this decay can only take place via a loop diagram.



Since the diagram involves two additional vertices and one additional loop integral, one might expect that the amplitude is down by a factor  $g^2/16\pi^2 = \alpha_W/4\pi$  where  $\alpha_W = g^2/4\pi = 1/29$  is the weak analog of the fine structure constant.<sup>†</sup> The branching ratio should then be down by a factor  $(\alpha_W/4\pi)^2 \sim 10^{-5}$ . But current measurements give a branching ratio, summed over neutrino flavors, of

$$\text{BR}(K^+ \rightarrow \pi^+ \bar{\nu} \nu) = 1.5^{+1.3}_{-0.9} \times 10^{-10}.$$

Clearly some additional suppression is called for. This is provided by the GIM mechanism. To see how it works, look at the contribution to the amplitude coming from the lower quark line.

$$\sum_{i=u,c,t} \int \frac{d^4 p}{(2\pi)^4} \bar{v}(s) \left( -\frac{ig}{2\sqrt{2}} \gamma^\mu (1 - \gamma^5) (V^\dagger)_{si} \right) \frac{i}{\not{p} - m_i} \left( -\frac{ig_Z}{2} (c_{Vi} \gamma^\nu - c_{Ai} \gamma^\nu \gamma^5) \right) \frac{i}{\not{p} - \not{q} - m_i} \left( -\frac{ig}{2\sqrt{2}} \gamma^\lambda (1 - \gamma^5) V_{id} \right) v(d)$$

If the quark masses were equal this would be proportional to  $\sum_i (V^\dagger)_{si} V_{id}$ , which vanishes by unitarity of the CKM matrix. More generally the amplitude picks up a GIM suppression factor,  $\mathcal{M} \sim \sum_i (V^\dagger)_{si} V_{id} f(m_i^2/m_W^2)$ . One can estimate the behavior of the function  $f(x)$  by expanding the quark propagator in powers of the quark mass,

$$\frac{1}{\not{p} - m_i} = \frac{\not{p}}{p^2} + \frac{m_i}{p^2} + \frac{m_i^2 \not{p}}{p^4} + \dots$$

The zeroth order terms all cancel, since they correspond to having equal (vanishing) quark masses. The first order terms vanish by chirality, since

<sup>†</sup> A factor  $1/16\pi^2$  is usually associated with each loop integral, as discussed on p. 102.



$(1 - \gamma^5)(\text{odd } \# \gamma\text{'s})(1 - \gamma^5) = 0$ . The second order terms leave you with a GIM suppression factor  $\sum_i (V^\dagger)_{si} V_{id} m_i^2 / m_W^2$ .<sup>‡</sup> This factor suppresses the contributions of the up and charm quarks; the top quark contribution is suppressed by the small mixing with the third generation. Taking all this into account leads to the current theoretical estimate for the branching ratio,<sup>§</sup>

$$\text{BR}(K^+ \rightarrow \pi^+ \bar{\nu} \nu) = (8.4 \pm 1.0) \times 10^{-11}.$$

This might seem like a remarkable but obscure prediction of the standard model. But experimental tests of this small branching ratio have far-reaching consequences. Almost every extension of the standard model introduces new sources of flavor violation, which could easily overwhelm the tiny standard model prediction. So limits on rare flavor-violating processes provide some of the most stringent constraints on beyond-the-standard model physics.

### 14.5 *CP violation*

As we've seen the CKM matrix involves three mixing angles between the different generations plus one complex phase. The complex phase turns out to be the only source for *CP* violation in the standard model.<sup>†</sup> It's remarkable that with two generations the considerations of section 12.3 would show that the CKM matrix is real: a  $2 \times 2$  orthogonal matrix parametrized by the Cabibbo angle. So in this sense *CP* violation is a bonus feature of the standard model associated with having three generations.

To see that *CP* is violated consider a tree-level decay  $u_i \rightarrow d_j W^+$ . Both the up-type and down-type quarks  $u_i, d_j$  must sit in left-handed spinors to couple to the  $W$ . Neglecting quark masses for simplicity, this means they're both left-handed particles. So indicating helicity with a subscript, we can denote this decay  $u_{Li} \rightarrow d_{Lj} W^+$ . Under a parity transformation the quark momentum changes sign, while the quark spin is invariant, so the helicity flips and the parity-transformed process is

$$P : u_{Ri} \rightarrow d_{Rj} W^+.$$

This decay doesn't occur at tree level, which should be no surprise – parity is maximally violated by the weak interactions. Charge conjugation exchanges

<sup>‡</sup> One also has to worry about the divergence structure of the remaining loop integral, which in the case at hand turns out to give an extra factor of  $\log m_i^2 / m_W^2$ .

<sup>§</sup> The theoretical status is reviewed in C. Smith, arXiv:hep-ph/0703039.

<sup>†</sup> Leaving aside a topological term built from the gluon field strength  $\epsilon^{\mu\nu\lambda\sigma} \text{Tr}(G_{\mu\nu} G_{\lambda\sigma})$  that can be added to the QCD Lagrangian.

particles with antiparticles while leaving helicity unchanged. So the charge conjugate of our original process is

$$C : \bar{u}_{Li} \rightarrow \bar{d}_{Lj} W^- .$$

But this decay doesn't occur at tree level either (the left-handed antiparticles sit in right-handed spinors which don't couple to the  $W$ ). Again, no surprise, since  $C$  is also maximally violated by the weak interactions. If we apply the combined transformation  $CP$  we get an allowed tree-level process,  $\bar{u}_{Ri} \rightarrow \bar{d}_{Rj} W^-$ . Does this mean  $CP$  is a symmetry? Compare the amplitudes:

$$\begin{aligned} u_{Li} \rightarrow d_{Lj} W^+ &\sim (V^\dagger)_{ji} = V_{ij}^* \\ \bar{u}_{Ri} \rightarrow \bar{d}_{Rj} W^- &\sim V_{ij} \end{aligned}$$

If the CKM matrix is not real then  $CP$  is violated.<sup>†</sup> One can reach the same conclusion, of course, by studying how  $CP$  acts on the standard model Lagrangian.

The classic evidence for  $CP$  violation comes from the neutral kaon system. The strong-interaction eigenstates  $K^0$ ,  $\bar{K}^0$  transform into each other under  $CP$ .

$$CP|K^0\rangle = |\bar{K}^0\rangle \quad CP|\bar{K}^0\rangle = |K^0\rangle$$

We can form  $CP$  eigenstates

$$\begin{aligned} |K_{\text{even}}\rangle &= \frac{1}{\sqrt{2}} (|K^0\rangle + |\bar{K}^0\rangle) \\ |K_{\text{odd}}\rangle &= \frac{1}{\sqrt{2}} (|K^0\rangle - |\bar{K}^0\rangle) . \end{aligned}$$

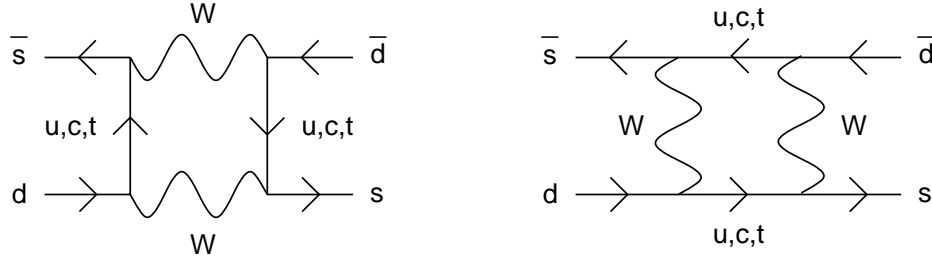
<sup>†</sup> For the particular process we are considering we could redefine the phases of our initial and final states to make the amplitude real. To have observable  $CP$  violation all three generations of quarks must be involved so the complex phase can't be eliminated by a field redefinition. Also more than one diagram must contribute, so that relative phases of diagrams can be observed through interference.

If there were no  $CP$  violation then  $|K_{\text{even}}\rangle$  and  $|K_{\text{odd}}\rangle$  would be the exact mass eigenstates. But the weak interactions which generate mixing between  $K^0$  and  $\bar{K}^0$  violate  $CP$ . The actual mass eigenstates  $K_L^0$ ,  $K_S^0$  are not  $CP$  eigenstates, as shown by the fact that  $K_L^0$  decays to both  $2\pi$  and  $3\pi$  final states ( $CP$  even and odd, respectively) with branching ratios

$$\text{BR}(K_L^0 \rightarrow \pi\pi) = 3.0 \times 10^{-3}$$

$$\text{BR}(K_L^0 \rightarrow \pi\pi\pi) = 34\%$$

This mixing gives rise to a mass splitting  $m_{K_L^0} - m_{K_S^0} = 3.5 \times 10^{-6} \text{ eV}$ . This is another example of a GIM-suppressed quantity, as one can see by examining the diagrams responsible for the mixing:



With equal quark masses the diagram would vanish by unitarity of the CKM matrix.

## 14.6 Custodial $SU(2)$

The sector of the standard model which is least satisfactory (from a theoretical point of view) and least well-tested (from an experimental point of view) is the sector associated with electroweak symmetry breaking. In the standard model the Higgs doublet seems put in by hand, for no other reason than to break electroweak symmetry, and we have no direct experimental evidence that a physical Higgs particle exists. You might think the only thing we know for sure is that the gauge symmetry is broken from  $SU(2)_L \times U(1)_Y$  to  $U(1)_{\text{em}}$ , with three would-be Goldstone bosons that get eaten to become the longitudinal polarizations of the  $W$  and  $Z$  bosons.

This is a little too pessimistic: there are some robust statements we can make about the nature of electroweak symmetry breaking. To see this it's useful to begin by rewriting the Higgs Lagrangian. Normally, neglecting all gauge couplings, we'd write the pure Higgs sector of the standard model in

terms of an  $SU(2)_L$  doublet  $\phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix}$ .

$$\mathcal{L}_{\text{pure Higgs}} = \partial_\mu \phi^\dagger \partial^\mu \phi + \mu^2 \phi^\dagger \phi - \lambda (\phi^\dagger \phi)^2$$

The conjugate Higgs doublet is defined by  $\tilde{\phi} = \epsilon \phi^* = \begin{pmatrix} \phi^{0*} \\ -\phi^- \end{pmatrix}$ . Although  $\tilde{\phi}$  is not an independent field, it's useful to treat  $\phi$  and  $\tilde{\phi}$  on the same footing. To do this we define a  $2 \times 2$  complex matrix

$$\Sigma = \sqrt{2} \begin{pmatrix} \tilde{\phi}, \phi \end{pmatrix} = \sqrt{2} \begin{pmatrix} \phi^{0*} & \phi^+ \\ -\phi^- & \phi^0 \end{pmatrix}.$$

This matrix satisfies

$$\Sigma^\dagger \Sigma = 2\phi^\dagger \phi \mathbf{1} \quad \det \Sigma = 2\phi^\dagger \phi \quad \Sigma^* = \sigma^2 \Sigma \sigma^2$$

(the last relation is a “pseudo-reality condition”). In any case, in terms of  $\Sigma$ , the pure Higgs Lagrangian is

$$\mathcal{L}_{\text{pure Higgs}} = \frac{1}{4} \text{Tr} \left( \partial_\mu \Sigma^\dagger \partial^\mu \Sigma \right) + \frac{1}{4} \mu^2 \text{Tr} \left( \Sigma^\dagger \Sigma \right) - \frac{1}{16} \lambda \left( \text{Tr}(\Sigma^\dagger \Sigma) \right)^2.$$

Written in this way it's clear that  $\mathcal{L}_{\text{pure Higgs}}$  has an  $SU(2)_L \times SU(2)_R$  global symmetry which acts on  $\Sigma$  as  $\Sigma \rightarrow L \Sigma R^\dagger$  for  $L, R \in SU(2)$ . In fact  $\mathcal{L}_{\text{pure Higgs}}$  is nothing but the  $O(4)$  linear  $\sigma$ -model from problem 5.4! The curious fact is that  $\mathcal{L}_{\text{pure Higgs}}$  has a larger symmetry group than is strictly necessary – larger, that is, than the  $SU(2)_L \times U(1)_Y$  gauge symmetry of the standard model.

Let's proceed to couple  $\mathcal{L}_{\text{pure Higgs}}$  to the electroweak gauge fields. An  $SU(2)_L \times U(1)_Y$  gauge transformation of  $\phi$ ,

$$\phi(x) \rightarrow e^{-ig\alpha^a(x)\sigma^a/2} e^{-ig'\alpha(x)/2} \phi(x),$$

corresponds to the following transformation of  $\Sigma$ .

$$\Sigma(x) \rightarrow e^{-ig\alpha^a(x)\sigma^a/2} \Sigma(x) e^{ig'\alpha(x)\sigma^3/2}$$

This shows that the  $SU(2)_L$  gauge symmetry of the standard model is identified with the  $SU(2)_L$  symmetry of  $\mathcal{L}_{\text{pure Higgs}}$ , while  $U(1)_Y$  is embedded as a subgroup of  $SU(2)_R$ . The covariant derivative becomes

$$\mathcal{D}_\mu \Sigma = \partial_\mu \Sigma + \frac{ig}{2} W_\mu^a \sigma^a \Sigma - \frac{ig'}{2} \Sigma B_\mu \sigma^3.$$

In this notation the Higgs sector of the standard model is

$$\mathcal{L}_{\text{Higgs}} = \frac{1}{4} \text{Tr} \left( \mathcal{D}_\mu \Sigma^\dagger \mathcal{D}^\mu \Sigma \right) + \frac{1}{4} \mu^2 \text{Tr} \left( \Sigma^\dagger \Sigma \right) - \frac{1}{16} \lambda \left( \text{Tr}(\Sigma^\dagger \Sigma) \right)^2.$$

The analysis of electroweak symmetry breaking is straightforward: the Higgs potential is minimized when  $\phi^\dagger \phi = \frac{1}{2}v^2$ , or equivalently when  $\Sigma^\dagger \Sigma = v^2 \mathbf{1}$ . The space of vacua is given by

$$\{\Sigma = vU : U \in SU(2)\}.$$

All these vacua are gauge-equivalent. Choosing any particular vacuum breaks  $SU(2)_L \times U(1)_Y \rightarrow U(1)_{\text{em}}$ .

The interesting observation is that the pure Higgs sector of the standard model has a larger symmetry than required for gauge invariance. The extra  $SU(2)_R$  symmetry of the pure Higgs Lagrangian is known as “custodial  $SU(2)$ ”.<sup>†</sup> It is *not* a symmetry of the entire standard model – it’s broken explicitly by the couplings of the hypercharge gauge boson, which pick out a  $U(1)_Y$  subgroup of  $SU(2)_R$ , as well as by the quark Yukawa couplings.

Despite this explicit breaking, custodial  $SU(2)$  has observable consequences. In particular, as you’ll show on the homework, it enforces the tree-level relation

$$\rho \equiv \frac{m_W^2}{m_Z^2 \cos^2 \theta_W} = 1.$$

The observed value is<sup>‡</sup>

$$\rho = 1.0106 \pm 0.0006.$$

The fact that the tree-level relation is satisfied to roughly 1% accuracy is strong evidence that the mechanism for electroweak symmetry breaking must have a custodial  $SU(2)$  symmetry. (Small deviations from  $\rho = 1$  can be understood as arising from radiative corrections in the standard model.)

To appreciate these statements let’s be completely general in our approach to electroweak symmetry breaking. We don’t really know that the standard model Higgs doublet exists, but we are certain that the gauge symmetry is broken. On general grounds there must be three would-be Goldstone bosons that get eaten to provide the longitudinal polarizations of the  $W$  and  $Z$  bosons. The Goldstones can be packaged into a matrix  $U \in SU(2)$ .<sup>§</sup> Up to two derivatives, the most general action for the Goldstones with  $SU(2)_L \times U(1)_Y$  symmetry is

$$\mathcal{L}_{\text{Goldstone}} = \frac{1}{4}v^2 \text{Tr} \left( \mathcal{D}_\mu U^\dagger \mathcal{D}^\mu U \right) + \frac{1}{4}c v^2 \text{Tr} \left( U^\dagger \mathcal{D}_\mu U \sigma^3 \right) \text{Tr} \left( U^\dagger \mathcal{D}^\mu U \sigma^3 \right).$$

<sup>†</sup> Note that some authors use the term custodial  $SU(2)$  to refer to the diagonal subgroup of  $SU(2)_L \times SU(2)_R$ .

<sup>‡</sup> The quoted value is for the quantity denoted  $\hat{\rho}$  in the particle data book.

<sup>§</sup> The space of vacua is  $(SU(2)_L \times U(1)_Y) / U(1)_{\text{em}}$ , which is topologically a three dimensional sphere. Points on a 3-sphere can be labeled by  $SU(2)$  matrices.

Here  $v$  and  $c$  are constants and the covariant derivative is

$$\mathcal{D}_\mu U = \partial_\mu U + \frac{ig}{2} W_\mu^a \sigma^a U - \frac{ig'}{2} U B_\mu \sigma^3.$$

The first term in the Lagrangian is exactly what we get from the standard model by setting  $\Sigma = vU$  in  $\mathcal{L}_{\text{Higgs}}$ . It has custodial  $SU(2)$  symmetry if you neglect the hypercharge gauge boson. The second term in the Lagrangian violates custodial  $SU(2)$ . In the standard model it only arises from operators of dimension 6 or higher; for this reason custodial  $SU(2)$  should be regarded as an accidental symmetry of the standard model. The key point is that in a model for electroweak symmetry breaking without custodial  $SU(2)$  one would expect  $c$  to be  $\mathcal{O}(1)$ . This would make  $\mathcal{O}(1)$  corrections to the relation  $\rho = 1$ , in drastic conflict with observation.

## References

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*CP* VIOLATION. Cheng & Li discuss *CP* violation in the kaon system in section 12.2. *CP* violation in the *B* mesons is covered in the review articles hep-ph/0411138 and hep-ph/0410351.

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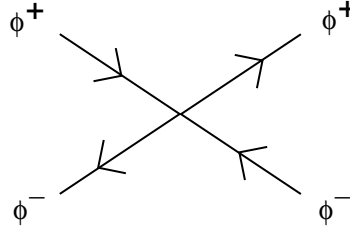
*S* AND *T* PARAMETERS. The *S* and *T* parameters were introduced by Peskin and Takeuchi, *Phys. Rev.* **D46** (1992) 381. They have been discussed in terms of effective field theory in many places. T. Appelquist and G.-H. Wu, arXiv:hep-ph/9304240, relate them to parameters in the effective Lagrangian for the electroweak Goldstone bosons. C.P. Burgess, S. Godfrey, H. König, D. London and I. Maksymyk, arXiv:hep-ph/9312291 give a procedure for relating the effective Lagrangian to observable quantities. The state of the art in this sort of analysis can be found in Z. Han and W. Skiba, arXiv:hep-ph/0412166. The current experimental bounds on *S* and *T* are given in Fig. 8 of J. Erler, arXiv:hep-ph/0604035.

## Exercises

### 14.1 Unitarity made easy

At very high energies it shouldn't matter whether the standard model gauge symmetry is broken or unbroken. Suppose it's unbroken (if you like, take  $\mu^2 < 0$  in the Higgs potential).

- (i) Use tree-level unitarity to bound the Higgs coupling  $\lambda$  by considering  $\phi^+ - \phi^-$  scattering coming from the diagram



Here we're writing the Higgs doublet as  $\phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix}$  with  $\phi^- =$

$(\phi^+)^*$ . Other diagrams contribute, but this one dominates for large  $\lambda$ .

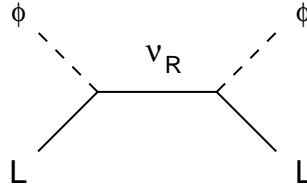
- (ii) Is your result equivalent to the bound on the Higgs mass (14.2) we obtained from studying  $W_L^+ W_L^- \rightarrow W_L^+ W_L^-$ ?

## 14.2 See-saw mechanism

There's an appealing extension of the standard model which generates small Majorana neutrino masses. Introduce a collection of  $n_s$  right-handed neutrinos, described by a collection of right-handed gauge singlet spinor fields  $\nu_{Ra}$ ,  $a = 1, \dots, n_s$  ( $s$  stands for sterile). Since these fields are gauge singlets they can have Majorana mass terms. The general renormalizable, gauge-invariant Lagrangian describing the left- and right-handed neutrinos and their couplings to the Higgs is then

$$\mathcal{L}_\nu = -\frac{1}{2} M_{ab} \overline{\nu_{RaC}} \nu_{Rj} - \Lambda_{ia}^\nu \bar{L}_i \tilde{\phi} \nu_{Ra} + \text{c.c.}$$

Let's imagine that the right-handed neutrinos are very heavy, with  $M_{ab} \gg v$  (there's no reason for  $M_{ab}$  to be tied to the electroweak symmetry breaking scale). Use the equations of motion for the right-handed neutrinos  $\frac{\partial \mathcal{L}_\nu}{\partial \nu_{Ra}} = 0$  to write down a low-energy effective Lagrangian involving just the Higgs field and the left-handed doublets. If you want to think in terms of Feynman diagrams, this is equivalent to evaluating the diagram



where we're neglecting the momentum dependence of the right-handed neutrino propagator. Show that this procedure induces precisely the operator (14.5), and read off the mass matrix for the left-handed neutrinos. This is known as the see-saw mechanism: the heavier the right-handed neutrinos are, the lighter the left-handed neutrinos become.



### 14.3 Custodial $SU(2)$ and the $\rho$ parameter

Consider the most general two-derivative action for the Goldstone bosons associated with electroweak symmetry breaking.

$$\mathcal{L} = \frac{1}{4}v^2 \text{Tr} \left( \mathcal{D}_\mu U^\dagger \mathcal{D}^\mu U \right) + \frac{1}{4}cv^2 \text{Tr} \left( U^\dagger \mathcal{D}_\mu U \sigma^3 \right) \text{Tr} \left( U^\dagger \mathcal{D}^\mu U \sigma^3 \right)$$

Here  $U \in SU(2)$  is the field describing the Goldstones, with covariant derivative

$$\mathcal{D}_\mu U = \partial_\mu U + \frac{ig}{2} W_\mu^a \sigma^a U - \frac{ig'}{2} U B_\mu \sigma^3.$$

Evaluate the  $W$  and  $Z$  masses in this model. Express the “ $\rho$  parameter”  $\rho = \frac{m_W^2}{m_Z^2 \cos^2 \theta_W}$  in terms of  $c$ .

### 14.4 Quark masses and custodial $SU(2)$ violation

Suppose we require that custodial  $SU(2)$  be a symmetry of the quark Yukawa Lagrangian

$$\mathcal{L}_{\text{quark Yukawa}} = -\Lambda_{ij}^d \bar{Q}_i \phi d_{Rj} - \Lambda_{ij}^u \bar{Q}_i \tilde{\phi} u_{Rj} + \text{c.c.}$$

What would this imply about the spectrum of quark masses? What would this imply about the CKM matrix?

### 14.5 Strong interactions and electroweak symmetry breaking

Consider a theory which resembles the standard model in every respect except that it doesn't have a Higgs field. You can get the Lagrangian for this theory by setting  $\phi = 0$  in the standard model Lagrangian; the Dirac and Yang-Mills terms survive while the Higgs and Yukawa terms drop out. In such a theory, what are the masses of the  $W$  and  $Z$  bosons?

Before your answer “zero,” recall the effective Lagrangian for chiral symmetry breaking by the strong interactions,  $\mathcal{L} = \frac{1}{4}f^2 \text{Tr} (\partial_\mu U^\dagger \partial^\mu U)$ . Let's concentrate on the up and down quarks so that  $U$  is an  $SU(2)$  matrix. As discussed in chapter 6 it's related to the chiral condensate by

$$\langle 0 | \psi_L \bar{\psi}_R | 0 \rangle = \mu^3 U \otimes \frac{1}{2}(1 - \gamma^5) \quad \psi = \begin{pmatrix} u \\ d \end{pmatrix}$$

where we've indicated the flavor  $\otimes$  spin structure of the condensate on the right hand side.

- (i) By introducing a suitable covariant derivative, write the effective Lagrangian which describes the couplings between  $U$  and the  $SU(2)_L \times U(1)_Y$  gauge fields  $W_\mu, B_\mu$ .

- (ii) Compute the  $W$  and  $Z$  masses in terms of  $f$  and the  $SU(2)_L \times U(1)_Y$  gauge couplings  $g, g'$ .
- (iii) In this model, what particles get eaten to give the  $W$  and  $Z$  bosons a mass?
- (iv) Does this model have a custodial  $SU(2)$  symmetry?
- (v) In the real world, taking both the Higgs field and QCD effects into account, what are the masses of the  $W$  and  $Z$  bosons? Express your answer in terms of  $f, g, g'$  and the Higgs vev  $v$ . Hint: think in terms of effective Lagrangians for the would-be Goldstone bosons.

This sort of idea – generating masses from underlying strongly-coupled gauge dynamics – is the basis for what are known as technicolor models of electroweak symmetry breaking.

#### 14.6 $S$ and $T$ parameters

A useful way to think about beyond-the-standard-model physics is to encode the effects of any new physics in the coefficients of higher-dimension operators which are added to the standard model Lagrangian. At dimension 5 there's a unique operator one can add which we encountered when we discussed neutrino masses. At dimension 6 there are quite a few possible operators.<sup>†</sup> The two which have attracted the most attention correspond to the  $S$  and  $T$  parameters of Peskin and Takeuchi. They can be defined by the dimension 6 Lagrangian

$$\mathcal{L}_{\text{dim } 6} = \frac{gg'}{16\pi v^2} S \phi^\dagger \sigma^a \phi W_{\mu\nu}^a B^{\mu\nu} - \frac{2\alpha}{v^2} T (\phi^\dagger \mathcal{D}_\mu \phi) (\mathcal{D}^\mu \phi^\dagger \phi).$$

Here  $g$  and  $g'$  are standard model gauge couplings,  $v$  is the Higgs vev,  $\alpha$  is the fine structure constant,  $\phi$  is the Higgs doublet,  $W_{\mu\nu}^a$  and  $B_{\mu\nu}$  are field strengths, and the constants  $S$  and  $T$  parametrize new physics.

- (i) In the notation of section 14.6 a particular vacuum state can be characterized by  $\Sigma = vU$  for some  $U \in SU(2)$ . Evaluate  $\mathcal{L}_{\text{dim } 6}$  at low energies and show that it reduces to

$$-\frac{gg'}{64\pi} S \text{Tr} (U^\dagger \sigma^a U \sigma^3) W_{\mu\nu}^a B^{\mu\nu} + \frac{1}{8} \alpha v^2 T \text{Tr} (U^\dagger \mathcal{D}_\mu U \sigma^3) \text{Tr} (U^\dagger \mathcal{D}^\mu U \sigma^3).$$

- (ii) In unitary gauge one conventionally sets  $U = \mathbf{1}$ . Evaluate your

<sup>†</sup> They're classified in W. Buchmuller and D. Wyler, *Nucl. Phys.* **B268** (1986) 621.

result from part (i) in unitary gauge and show that to quadratic order in the fields it reduces to

$$-\frac{1}{8}\alpha S \left( F_{\mu\nu}F^{\mu\nu} - Z_{\mu\nu}Z^{\mu\nu} + \frac{\cos^2\theta_W - \sin^2\theta_W}{\cos\theta_W \sin\theta_W} F_{\mu\nu}Z^{\mu\nu} \right) - \frac{1}{2}\alpha m_Z^2 Z_\mu Z^\mu.$$

Here  $F_{\mu\nu}$  is the field strength of electromagnetism and  $Z_{\mu\nu}$  is the abelian field strength associated with the  $Z$  boson. The fields  $A_\mu$  and  $Z_\mu$  have their usual standard model definitions; note that when  $\mathcal{L}_{\text{dim } 6}$  is added they no longer have canonical kinetic terms. Also  $m_Z$  is the usual standard model definition of the  $Z$  mass; note that when  $\mathcal{L}_{\text{dim } 6}$  is added it no longer corresponds to the physical  $Z$  mass.

#### 14.7 $B - L$ as a gauge symmetry

The standard model has an accidental global symmetry corresponding to conservation of  $B - L$ . This symmetry can be gauged as follows. Consider the electroweak interactions of a single generation of quarks and leptons and promote the gauge symmetry to  $SU(2)_L \times U(1)_Y \times U(1)_{B-L}$ . To the usual standard model fields add a right-handed neutrino  $\nu_R$  and a complex scalar field  $\chi$ . Overall we have fields with quantum numbers

$L$	$(\mathbf{2}, -1, -1)$
$e_R$	$(\mathbf{1}, -2, -1)$
$Q$	$(\mathbf{2}, 1/3, 1/3)$
$u_R$	$(\mathbf{1}, 4/3, 1/3)$
$d_R$	$(\mathbf{1}, -2/3, 1/3)$
$\phi$	$(\mathbf{2}, 1, 0)$
$\tilde{\phi}$	$(\mathbf{2}, -1, 0)$
$\nu_R$	$(\mathbf{1}, 0, -1)$
$\chi$	$(\mathbf{1}, 0, 1)$

So for instance the covariant derivative of  $\nu_R$  is

$$\mathcal{D}_\mu \nu_R = \partial_\mu \nu_R + i\tilde{g}(-1)C_\mu \nu_R$$

where  $C_\mu$  is the  $U(1)_{B-L}$  gauge field and  $\tilde{g}$  is its coupling constant.

- (i) Show that, thanks to  $\nu_R$ , the  $U(1)_{B-L}$  symmetry is anomaly-free.
- (ii) If left unbroken  $U(1)_{B-L}$  would mediate a Coulomb-like force with  $B - L$  playing the role of electric charge. To cure this suppose the Higgs Lagrangian

$$\mathcal{L}_{\text{Higgs}} = \mathcal{D}_\mu \phi^\dagger \mathcal{D}^\mu \phi + \mathcal{D}_\mu \chi^* \mathcal{D}^\mu \chi - V(\phi^\dagger \phi, \chi^* \chi)$$

is such that the scalar fields acquire vevs.

$$\langle 0|\phi|0\rangle = \begin{pmatrix} 0 \\ v/\sqrt{2} \end{pmatrix} \quad \langle 0|\chi|0\rangle = \tilde{v}/\sqrt{2}$$

We'll have in mind that  $\tilde{v} \gg v$ . Add additional kinetic terms to the Yang-Mills Lagrangian:

$$\mathcal{L}_{\text{Yang-Mills}} = (\text{standard model}) - \frac{1}{4}C_{\mu\nu}C^{\mu\nu} - \frac{1}{2}aB_{\mu\nu}C^{\mu\nu}$$

Here  $C_{\mu\nu}$  is the  $U(1)_{B-L}$  field strength. The parameter  $a$  represents kinetic mixing between the hypercharge and  $B-L$  gauge fields. Expand about the vacuum state and write down the quadratic action for the gauge bosons  $W_\mu^3, B_\mu, C_\mu$ .

- (iii) To have positive-definite kinetic terms we must have  $-1 < a < 1$ . Set  $a = \sin \alpha$  and show that the kinetic terms can be diagonalized by setting

$$\begin{pmatrix} W_\mu^3 \\ B_\mu \\ C_\mu \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -\tan \alpha \\ 0 & 0 & 1/\cos \alpha \end{pmatrix} \begin{pmatrix} \hat{W}_\mu^3 \\ \hat{B}_\mu \\ \hat{C}_\mu \end{pmatrix}$$

Show that the mass matrix can then be diagonalized by setting

$$\begin{pmatrix} \hat{A}_\mu \\ \hat{Z}_\mu \\ \hat{C}_\mu \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \beta & \sin \beta \\ 0 & -\sin \beta & \cos \beta \end{pmatrix} \begin{pmatrix} A_\mu \\ Z_\mu \\ Z'_\mu \end{pmatrix}$$

where

$$\hat{A}_\mu = \frac{g'\hat{W}_\mu^3 + g\hat{B}_\mu}{\sqrt{g^2 + g'^2}} \quad \hat{Z}_\mu = \frac{g\hat{W}_\mu^3 - g'\hat{B}_\mu}{\sqrt{g^2 + g'^2}}$$

are the standard model photon and  $Z$ . Assuming  $\tilde{v} \gg v$  show that  $\beta \approx gg'v^2 \sin(2\alpha)/8\tilde{g}^2\tilde{v}^2$ .

- (iv) For  $\tilde{v} \gg v$  the mixing angle  $\beta$  is small. At leading order in  $\beta$  determine

- the masses of the  $A$ ,  $Z$  and  $Z'$  bosons
- the currents to which they couple

Are these results corrected at higher orders in  $\beta$ ?

Remarks: additional  $U(1)$  gauge groups arise in many extensions of the standard model. In general anomaly cancellation tightly constrains the allowed matter content and quantum numbers. Since  $B-L$  is an accidental symmetry of the standard model, it can be

spontaneously broken without significant observable consequences: there's no way to directly couple the symmetry-breaking vev  $\tilde{v}$  to the standard model at the renormalizeable level. At leading order in  $\beta$  the  $Z$  boson behaves just as in the standard model, but this gets corrected at higher orders. Finally in this model a term in the Yukawa Lagrangian  $-\lambda_\nu \bar{L} \tilde{\phi} \nu_R + \text{c.c.}$  could give neutrinos a large Dirac mass. Possible cures for this problem are reviewed in P. Langacker, *The physics of heavy  $Z'$  gauge bosons*, arXiv:0801.1345.

## 15

### Epilogue: in praise of the standard model

Physics 85200

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A good figure of merit for a theory is the ratio of results to assumptions. By this measure the standard model is impressive indeed. Given a fairly short list of assumptions – just the gauge group and matter content – the standard model is the most general renormalizable theory consistent with Lorentz and gauge invariance. The assumptions are open to criticism, for example

- the choice of gauge group seems a little peculiar
- the fermion representations are more complicated than one might have wished
- it's not clear why there should be three generations
- aside from simplicity, postulating a single Higgs doublet has very little motivation (experimental or otherwise)

Given these assumptions one gets a quite predictive theory. It depends on a total of 18 parameters.<sup>†</sup>

- 3 gauge couplings
- 2 parameters in the Higgs potential
- 9 quark and lepton masses
- 4 parameters in the CKM matrix

This isn't entirely satisfactory. The fermion masses and mixings, in particular, introduce more free parameters than one would like, and their observed values seem to exhibit peculiar hierarchies. The Higgs mass parameter  $\mu^2$  is also a puzzle. What sets its value? In fact, why should it be positive

<sup>†</sup> leaving aside certain topological terms in the action

(equivalently, why should electroweak gauge symmetry get broken)? But given these inputs look at what we get out:

- cancellation of gauge anomalies
- accidental baryon and lepton number conservation<sup>†</sup>
- accidental conservation of electron, muon, and tau number
- with three light quarks, an approximate  $SU(3)_L \times SU(3)_R$  symmetry of the strong interactions
- with a single Higgs doublet, a custodial  $SU(2)$  symmetry of the electroweak symmetry breaking sector
- massless neutrinos at the renormalizable level
- a natural explanation for small neutrino masses from dimension-5 operators
- absence of tree-level flavor changing neutral currents
- the GIM mechanism for suppressing flavor violation in loops
- with three generations, a mechanism for CP violation by the weak interactions

In the end, of course, the best thing about the standard model is that it fits the data. At low energies it incorporates all the successes of 4-Fermi theory and the  $SU(3)_L \times SU(3)_R$  symmetry of the strong interactions, and at high energies it fits the precision electroweak measurements carried out at LEP and SLC.

I wish I could say there was an extension of the standard model that was nearly as compelling as the standard model itself. Various extensions of the standard model have been proposed, each of which has some attractive features, but all of which have drawbacks. So far no one theory has emerged as a clear favorite. Only time, and perhaps the LHC, will tell us what lies beyond the standard model.

**Note added:** The previous paragraph was written in July 2007. In July 2012 the ATLAS and CMS experiments announced the discovery of a particle with a mass of 125 GeV which appears to be the Higgs boson predicted by the standard model: ATLAS Collaboration, *Observation of a new particle in the search for the standard model Higgs boson with the ATLAS detector at the LHC*, Phys. Lett. **B716**, 1 (2012), arXiv:1207.7214; CMS Collaboration, *Observation of a new boson at a mass of 125 GeV with the CMS experiment at the LHC*, Phys. Lett. **B716**, 30 (2012).

<sup>†</sup> strictly speaking  $B + L$  is violated by a quantum anomaly

# Appendix A

## Feynman diagrams

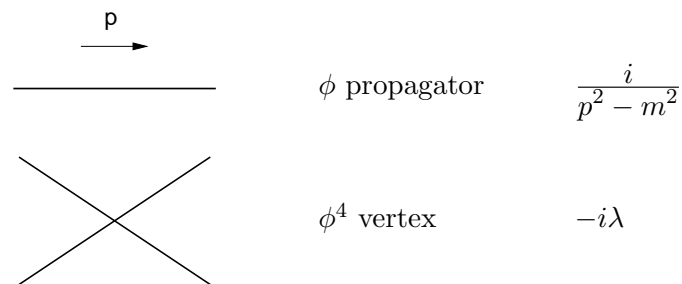
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Perhaps the simplest example is a real scalar field  $\phi$  with Lagrangian

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 - \frac{1}{4!} \lambda \phi^4.$$

In this case the Feynman rules are

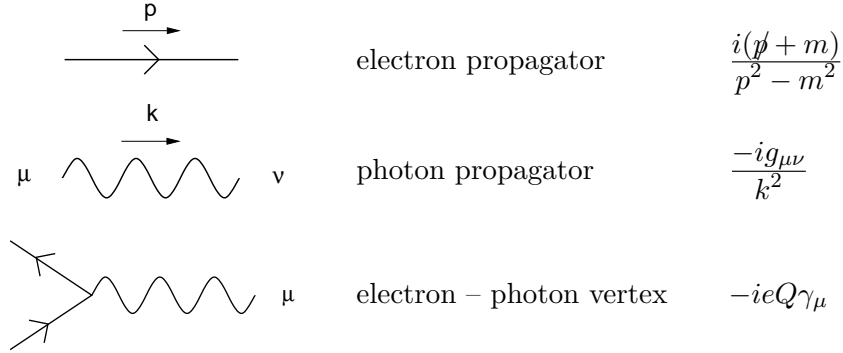


Here  $p$  is the 4-momentum flowing through the line. QED is somewhat more complicated: it's a Dirac spinor field coupled to a gauge field with Lagrangian

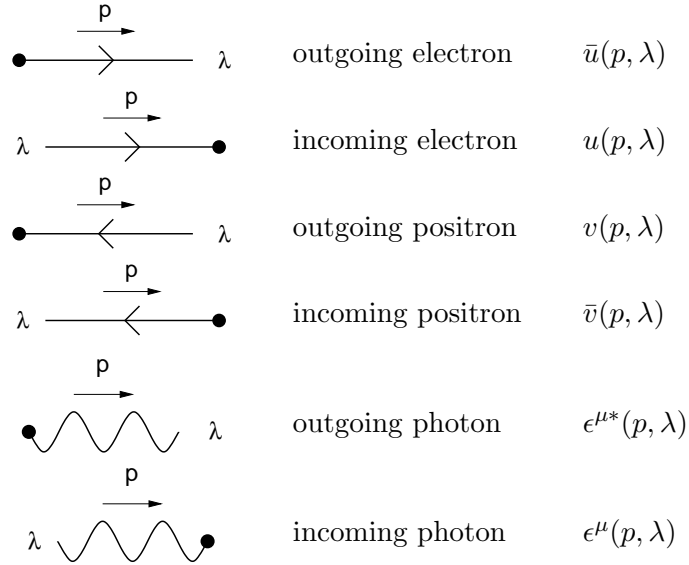
$$\mathcal{L} = \bar{\psi} \left[ i \gamma^\mu (\partial_\mu + ieQ A_\mu) - m \right] \psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}.$$

The corresponding Feynman rules are (with the notation  $\not{p} \equiv \gamma^\mu p_\mu$ )





Here  $Q$  is the charge of the field measured in units of  $e = \sqrt{4\pi\alpha}$ , for example  $Q = -1$  for the electron/positron field. The arrows on the lines indicate the direction of “particle flow,” for particles that have distinct antiparticles. We also have factors to indicate the polarizations of the external lines



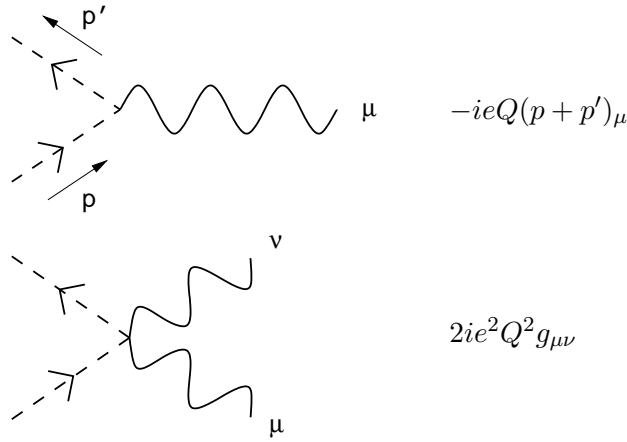
Here  $\lambda$  is a label that specifies the polarization of the particle. Explicit expressions are given in Peskin & Schroeder appendix A.2. Fortunately for most purposes all we’ll need are the completeness relations

$$\begin{aligned}
 \sum_{\lambda} u(p, \lambda) \bar{u}(p, \lambda) &= \not{p} + m \\
 \sum_{\lambda} v(p, \lambda) \bar{v}(p, \lambda) &= \not{p} - m \\
 \sum_{\lambda} \epsilon_{\mu}^*(p, \lambda) \epsilon_{\nu}(p, \lambda) &= -g_{\mu\nu}
 \end{aligned}$$

One can also consider the electrodynamics of a complex scalar field, with Lagrangian

$$\mathcal{L} = (\partial_\mu \phi^* - ieQA_\mu \phi^*) (\partial^\mu \phi + ieQA^\mu \phi) - m^2 \phi^* \phi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}.$$

In this case the interaction vertices are

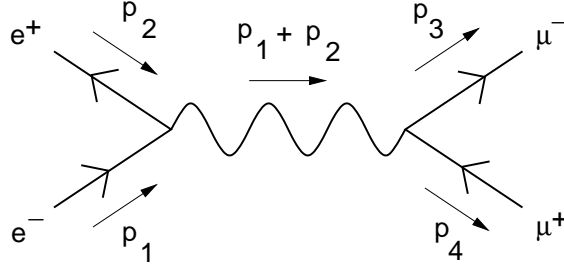


where the dashed line represents the scalar field.

In practice given a Lagrangian one can read off the Feynman rules as follows. First split the Lagrangian into free and interacting parts. The free part, which is necessarily quadratic in the fields, determines the propagators as described in section 9.2. Each term in the interacting part of the Lagrangian corresponds to a vertex, where the vertex factor can be obtained from the Lagrangian by the following recipe.

1. Erase the fields.
2. Multiply by  $i$ .
3. If there was a derivative operator  $\partial_\mu$  acting on a field, replace it with  $-ik_\mu$  where  $k_\mu$  is the incoming momentum of the corresponding line.
4. Multiply by  $s!$  for each group of  $s$  identical particles.

Given these rules it's just a matter of putting the pieces together: the sum of all Feynman diagrams gives  $-i$  times the amplitude  $\mathcal{M}$  for a given process. For example, for  $e^+e^- \rightarrow \mu^+\mu^-$  the lowest-order Feynman diagram is



Note that we're imposing 4-momentum conservation at every vertex. Working backwards along the fermion lines the diagram is equal to

$$-i\mathcal{M} = \bar{v}(p_2, \lambda_2)(-ieQ\gamma_\mu)u(p_1, \lambda_1)\frac{-ig^{\mu\nu}}{(p_1 + p_2)^2}\bar{u}(p_3, \lambda_3)(-ieQ\gamma_\nu)v(p_4, \lambda_4)$$

so that

$$\mathcal{M} = -\frac{e^2}{(p_1 + p_2)^2}\bar{v}(p_2, \lambda_2)\gamma_\mu u(p_1, \lambda_1)\bar{u}(p_3, \lambda_3)\gamma^\mu v(p_4, \lambda_4).$$

We're really interested in the transition probability, which is determined by (recall  $\bar{u} \equiv u^\dagger\gamma^0$ )

$$|\mathcal{M}|^2 = \frac{e^4}{(p_1 + p_2)^4}\bar{v}(p_2, \lambda_2)\gamma_\mu u(p_1, \lambda_1)u^\dagger(p_1, \lambda_1)\gamma_\nu^\dagger\gamma^{0\dagger}v(p_2, \lambda_2) \\ \bar{u}(p_3, \lambda_3)\gamma^\mu v(p_4, \lambda_4)v^\dagger(p_4, \lambda_4)\gamma^\nu\gamma^{0\dagger}u(p_3, \lambda_3)$$

We'll work in the chiral basis for the Dirac matrices, namely

$$\gamma^0 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$$

satisfying  $\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}$ ,  $\gamma^{0\dagger} = \gamma^0$  and  $\gamma^{i\dagger} = -\gamma^i$ . Note that  $\gamma^0\gamma^{\mu\dagger}\gamma^0 = \gamma^\mu$ . Then

$$|\mathcal{M}|^2 = \frac{e^4}{(p_1 + p_2)^4}\bar{v}(p_2, \lambda_2)\gamma_\mu u(p_1, \lambda_1)\bar{u}(p_1, \lambda_1)\gamma_\nu v(p_2, \lambda_2) \\ \bar{u}(p_3, \lambda_3)\gamma^\mu v(p_4, \lambda_4)\bar{v}(p_4, \lambda_4)\gamma^\nu u(p_3, \lambda_3) \\ = \frac{e^4}{(p_1 + p_2)^4}\text{Tr}(\gamma_\mu u(p_1, \lambda_1)\bar{u}(p_1, \lambda_1)\gamma_\nu v(p_2, \lambda_2)\bar{v}(p_2, \lambda_2)) \\ \text{Tr}(\gamma^\mu v(p_4, \lambda_4)\bar{v}(p_4, \lambda_4)\gamma^\nu u(p_3, \lambda_3)\bar{u}(p_3, \lambda_3))$$

If we're interested in unpolarized scattering we should average over initial spins and sum over final spins. This gives rise to the spin-averaged amplitude

$$\langle |\mathcal{M}|^2 \rangle = \frac{1}{4} \sum_{\lambda_1 \lambda_2 \lambda_3 \lambda_4} |\mathcal{M}|^2$$

$$= \frac{e^4}{4(p_1 + p_2)^4} \text{Tr}(\gamma_\mu(\not{p}_1 + m_e)\gamma_\nu(\not{p}_2 - m_e)) \text{Tr}(\gamma^\mu(\not{p}_4 - m_\mu)\gamma^\nu(\not{p}_3 + m_\mu))$$

where we've used the completeness relations to do the spin sums. For simplicity let's set  $m_e = m_\mu = 0$ , so that

$$\langle |\mathcal{M}|^2 \rangle = \frac{e^4}{4(p_1 + p_2)^4} \text{Tr}(\gamma_\mu \not{p}_1 \gamma_\nu \not{p}_2) \text{Tr}(\gamma^\mu \not{p}_4 \gamma^\nu \not{p}_3) .$$

Now we use the trace theorem

$$\text{Tr}(\gamma^\mu \gamma^\lambda \gamma^\nu \gamma^\sigma) = 4(g^{\mu\lambda} g^{\nu\sigma} - g^{\mu\nu} g^{\lambda\sigma} + g^{\mu\sigma} g^{\lambda\nu})$$

to get

$$\langle |\mathcal{M}|^2 \rangle = \frac{8e^4}{(p_1 + p_2)^4} (p_1 \cdot p_3 p_2 \cdot p_4 + p_1 \cdot p_4 p_2 \cdot p_3) .$$

With massless external particles momentum conservation  $p_1 + p_2 = p_3 + p_4$  implies that  $p_1 \cdot p_3 = p_2 \cdot p_4$  and  $p_1 \cdot p_4 = p_2 \cdot p_3$ , so

$$\langle |\mathcal{M}|^2 \rangle = \frac{8e^4}{(p_1 + p_2)^4} ((p_1 \cdot p_3)^2 + (p_1 \cdot p_4)^2) .$$

At this point one has to plug in some explicit kinematics. Let's work in the center of mass frame, with scattering angle  $\theta$ .

$$\begin{aligned} p_1 &= (E, 0, 0, E) \\ p_2 &= (E, 0, 0, -E) \\ p_3 &= (E, E \sin \theta, 0, E \cos \theta) \\ p_4 &= (E, -E \sin \theta, 0, -E \cos \theta) \end{aligned}$$

The spin-averaged amplitude is just

$$\langle |\mathcal{M}|^2 \rangle = e^4 (1 + \cos^2 \theta) .$$

We had to do a lot of work to get such a simple result! In general the center of mass differential cross section is given by

$$\left( \frac{d\sigma}{d\Omega} \right)_{\text{c.m.}} = \frac{1}{64\pi^2 s} \frac{|\vec{p}_3|}{|\vec{p}_1|} \langle |\mathcal{M}|^2 \rangle \quad (\text{A.1})$$

where  $s = (p_1 + p_2)^2$  and  $|\vec{p}_1|$ ,  $|\vec{p}_3|$  are the magnitudes of the *spatial* 3-momenta. So finally our differential cross section is

$$\left( \frac{d\sigma}{d\Omega} \right)_{\text{c.m.}} = \frac{e^4}{256\pi^2 E^2} (1 + \cos^2 \theta) .$$

The total cross section is given by integrating this over angles.

$$\sigma = 2\pi \int_{-1}^1 d(\cos \theta) \frac{d\sigma}{d\Omega} = \frac{e^4}{48\pi E^2}$$

This illustrates the basic process of evaluating a cross-section using Feynman diagrams. However there are a few subtle points that didn't come up in this simple example. In particular

- If there are undetermined internal loop momenta  $p$  in a diagram we should integrate over them with  $\int \frac{d^4 p}{(2\pi)^4}$ .
- If there are identical particles in the final state then (A.1) is still correct. However in computing the total cross section one should only integrate over angles corresponding to inequivalent final configurations. See Peskin & Schroeder p. 108.
- Every internal closed fermion loop multiplies the diagram by  $(-1)$ . This follows from Fermi statistics: Peskin & Schroeder p. 120.
- When summing diagrams there are sometimes relative  $(-)$  signs if the external lines obey Fermi statistics. See Griffiths pp. 231 and 235 or Peskin & Schroeder p. 119.
- In some cases diagrams have to be multiplied by combinatoric “symmetry factors.” These arise if a change in the internal lines of a diagram actually gives the same diagram back again. See Peskin & Schroeder p. 93.

## References

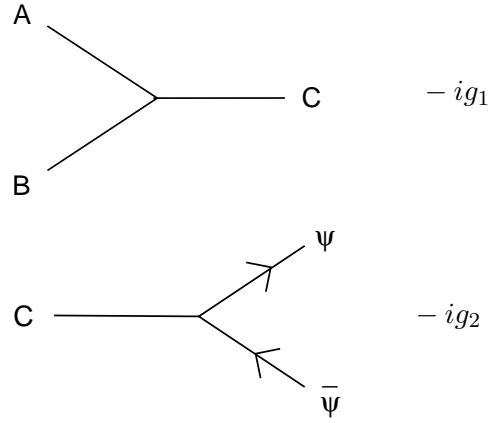
Griffiths does a nice job of presenting the Feynman rules for electrodynamics in sections 7.5 – 7.8. The process  $e^+e^- \rightarrow \mu^+\mu^-$  is studied in detail in Peskin & Schroeder section 5.1.

## Exercises

### A.1 $\text{ABC}\psi$ theory

Consider a theory with three real scalar fields  $A, B, C$  and one Dirac spinor field  $\psi$ . The masses of these particles are  $m_A, m_B, m_C, m_\psi$

and the interaction vertices are



- (i) Compute the partial width for the decay  $C \rightarrow \psi\bar{\psi}$ , assuming  $m_C > 2m_\psi$ .
- (ii) Compute the differential cross section for  $AB \rightarrow \psi\bar{\psi}$ .
- (iii) Find the total cross section for  $AB \rightarrow \psi\bar{\psi}$ .

# Appendix B

## Partial waves

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Consider a scattering process  $a + b \rightarrow c + d$ . Let's work in the center of mass frame, with initial and final states of definite helicity. We denote

$$\begin{aligned} E &= \text{total center of mass energy} \\ p &= \text{spatial momentum of either incoming particle} \\ \theta &= \text{center of mass scattering angle} \end{aligned}$$

That is, we take our incoming particles to have 4-momenta

$$p_a = (E_a, 0, 0, p) \quad p_b = (E_b, 0, 0, -p)$$

with  $E = E_a + E_b$ . We denote the helicities of the particles by  $\lambda_a, \lambda_b, \lambda_c, \lambda_d$ .

Our goal is to decompose the scattering amplitude into states of definite total angular momentum  $J$ . At first sight, this is a complicated problem: it seems we have to add the two spins plus whatever orbital angular momentum might be present. The analysis can be simplified by noting that the helicity ( $\equiv$  component of spin along the direction of motion) is a scalar quantity, invariant under spatial rotations. It therefore commutes with the total angular momentum. This means we can label our initial state

$$|E, J, J_z, \lambda_a, \lambda_b\rangle$$

by giving the center of mass energy  $E$ , the total angular momentum  $J$ , the  $z$  component  $J_z$ , and the two helicities  $\lambda_a, \lambda_b$ . In fact  $J_z$  is not an independent quantity. Our incoming particles have definite spatial momenta, described by wavefunctions

$$\psi_a \sim e^{ipz} \quad \psi_b \sim e^{-ipz}.$$

These wavefunctions are invariant under rotations in the  $xy$  plane, so the  $z$  component of the orbital angular momentum of the initial state vanishes,

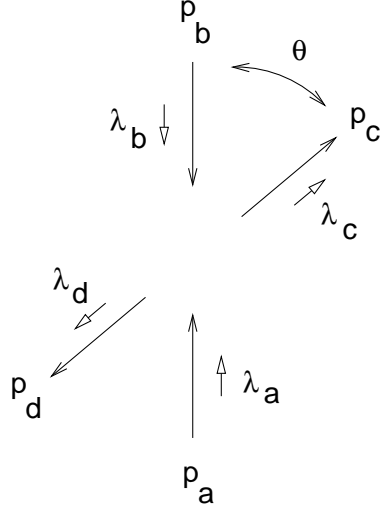


Fig. B.1. A picture of the reaction: spatial momenta indicated by large arrows, helicities indicated by small arrows.

$L_z = 0$ . This means  $J_z$  just comes from the helicities, and the initial state has  $J_z = \lambda_a - \lambda_b$ . Similar reasoning shows that we can label our final state by

$$|E, J, J_\theta, \lambda_c, \lambda_d\rangle.$$

Here  $E$  and  $J$  are the (conserved!) total energy and angular momentum of the system, while  $J_\theta$  is the component of  $J$  along the direction of particle  $c$ . As before we have  $J_\theta = \lambda_c - \lambda_d$ .

With these preliminaries in hand it's easy to determine the angular dependence of the scattering amplitude. The initial state can be regarded as an angular momentum eigenstate  $|J, J_z = \lambda\rangle$  where  $\lambda = \lambda_a - \lambda_b$ . The final state can be regarded as an eigenstate  $|J, J_\theta = \mu\rangle$  where  $\mu = \lambda_c - \lambda_d$ . We can make the final state by starting with a  $J_z$  eigenstate and applying a spatial rotation through an angle  $\theta$  about (say) the negative  $y$  axis.

$$|J, J_\theta = \mu\rangle = e^{i\theta\hat{J}_y}|J, J_z = \mu\rangle$$

Here  $\hat{J}_y$  is the  $y$  component of the angular momentum operator. The  $\theta$ -dependence of the scattering amplitude is given by the inner product of the initial and final states.

$$\begin{aligned} \mathcal{M} &\sim \langle J, J_\theta = \mu | J, J_z = \lambda \rangle \\ &= \langle J, J_z = \mu | e^{-i\theta\hat{J}_y} | J, J_z = \lambda \rangle \end{aligned}$$



$$\equiv d_{\mu\lambda}^J(\theta).$$

The quantity  $d_{\mu\lambda}^J(\theta)$  is known as a Wigner function: see Sakurai, *Modern quantum mechanics*, p. 192 – 195 and p. 221 – 223.

The angular dependence of a helicity amplitude is determined purely by group theory. To determine the overall coefficient one has to keep careful track of the normalization of the initial and final states. This was done by Jacob and Wick, who showed that the center-of-mass differential cross section is

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{c.m.}} = |f(\theta)|^2 \quad (\text{B.1})$$

where the scattering amplitude  $f(\theta)$  (normalized slightly differently from the usual relativistic scattering amplitude  $\mathcal{M}$ ) can be expanded in a sum of partial waves.

$$f(\theta) = \frac{1}{2i|\vec{p}|} \sum_{J=J_{\min}}^{\infty} (2J+1) \langle \lambda_c \lambda_d | S_J(E) - \mathbf{1} | \lambda_a \lambda_b \rangle d_{\mu\lambda}^J(\theta)$$

Here  $|\vec{p}|$  is the magnitude of the spatial momentum of either incoming particle. The sum over partial waves runs in integer steps starting from the minimum value  $J_{\min} = \max(|\lambda|, |\mu|)$ . The helicity states are unit-normalized,

$$\langle \lambda_a \lambda_b | \lambda'_a \lambda'_b \rangle = \delta_{\lambda_a \lambda'_a} \delta_{\lambda_b \lambda'_b} \quad \langle \lambda_c \lambda_d | \lambda'_c \lambda'_d \rangle = \delta_{\lambda_c \lambda'_c} \delta_{\lambda_d \lambda'_d}.$$

$S_J(E)$  is the  $S$ -matrix in the sector with total angular momentum  $J$  and total energy  $E$ . You only need to worry about subtracting off the identity operator if you're studying elastic scattering,  $a = c$  and  $b = d$ .

An important special case is when  $\lambda = \mu = 0$ , either because the incoming and outgoing particles are spinless, or because the initial and final states have no net helicity. In this case  $J$  is an integer and

$$d_{00}^J(\theta) = P_J(\cos \theta)$$

is a Legendre polynomial (Sakurai, *Modern quantum mechanics*, p. 202 – 203). The partial wave decomposition reduces to the familiar form

$$f(\theta) = \frac{1}{2i|\vec{p}|} \sum_{J=0}^{\infty} (2J+1) \langle \lambda_c \lambda_d | S_J(E) - \mathbf{1} | \lambda_a \lambda_b \rangle P_J(\cos \theta)$$

which is also valid in non-relativistic quantum mechanics. At high energies this goes over to the result (8.5) given in the text.

In general the Wigner functions satisfy an orthogonality relation

$$\int d\Omega d_{\mu\lambda}^J(\theta) \left(d_{\mu\lambda}^{J'}(\theta)\right)^* = \frac{4\pi}{2J+1} \delta_{JJ'}.$$

One can prove this along the lines of Georgi, *Lie algebras in particle physics*, section 1.12. Georgi's proof applies to finite groups, but the generalization to  $SU(2)$  is straightforward. (If you only want to check the coefficient, write the left hand side as

$$\int d\Omega \langle J, \mu | e^{-i\theta \hat{J}_y} | J, \lambda \rangle \langle J', \lambda | e^{i\theta \hat{J}_y} | J', \mu \rangle.$$

Set  $J = J'$ , sum over  $\lambda$ , and use  $\sum_{\lambda} |J, \lambda\rangle \langle J, \lambda| = \mathbf{1}$ .) Using this in (B.1) we can express the total cross section for scattering of distinguishable particles as

$$\sigma = \frac{\pi}{|\vec{p}|^2} \sum_{J=J_{\min}}^{\infty} (2J+1) \left| \langle \lambda_c \lambda_d | S_J(E) - \mathbf{1} | \lambda_a \lambda_b \rangle \right|^2.$$

As in chapter 8 we have a bound on the partial-wave cross sections for inelastic scattering, namely

$$\sigma = \sum_J \sigma_J \quad \text{with} \quad \sigma_J \leq \frac{\pi}{|\vec{p}|^2} (2J+1).$$

### References

The partial-wave expansion of a helicity amplitude was developed by M. Jacob and G. C. Wick, *Ann. Phys.* **7**, 404 (1959).

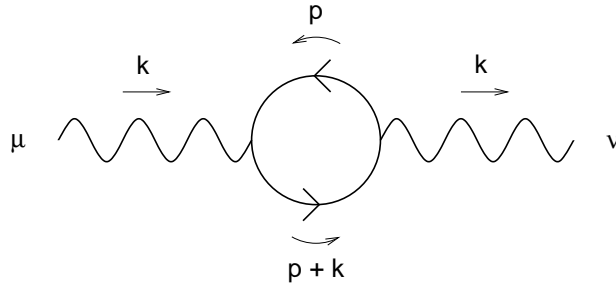
# Appendix C

## Vacuum polarization

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The purpose of this appendix is to study the one-loop QED vacuum polarization diagram



This diagram is discussed in every book on field theory. We'll evaluate it with a Euclidean momentum cutoff – an unusual approach, but one that provides an interesting contrast to the anomaly phenomenon discussed in chapter 13.

The basic amplitude is easy to write down.

$$-i\mathcal{M} = (-1) \int \frac{d^4p}{(2\pi)^4} \text{Tr} \left\{ (-ieQ\gamma^\mu) \frac{i(\not{p} + m)}{p^2 - m^2} (-ieQ\gamma^\nu) \frac{i(\not{p} + \not{k} + m)}{(p+k)^2 - m^2} \right\} \quad (\text{C.1})$$

(Recall that a closed fermion loop gives a factor of  $-1$ . Also note that we trace over the spinor indices that run around the loop.) Evaluating the trace

$$-i\mathcal{M} = -4e^2Q^2 \int \frac{d^4p}{(2\pi)^4} \frac{p^\mu(p+k)^\nu + (p+k)^\mu p^\nu - g^{\mu\nu}(p^2 + p \cdot k - m^2)}{(p^2 - m^2)((p+k)^2 - m^2)}.$$

Adopting a trick due to Feynman, we use the identity

$$\frac{1}{AB} = \int_0^1 \frac{dx}{(B + (A - B)x)^2}$$

to rewrite the amplitude as

$$-i\mathcal{M} = -4e^2Q^2 \int \frac{d^4p}{(2\pi)^4} \int_0^1 dx \frac{p^\mu(p+k)^\nu + (p+k)^\mu p^\nu - g^{\mu\nu}(p^2 + p \cdot k - m^2)}{(p^2 - m^2 + (k^2 + 2p \cdot k)x)^2}.$$

Now we change variables of integration from  $p^\mu$  to  $q^\mu = p^\mu + k^\mu x$ .<sup>†</sup>

$$-i\mathcal{M} = -4e^2Q^2 \int_0^1 dx \int \frac{d^4q}{(2\pi)^4} \frac{2q^\mu q^\nu - g^{\mu\nu}(q^2 - m^2) + (g^{\mu\nu}k^2 - 2k^\mu k^\nu)x(1-x) + (\text{odd in } q)}{(q^2 + k^2x(1-x) - m^2)^2}.$$

This might not seem like much of a simplification, but the beauty of Feynman's trick is that the denominator is Lorentz invariant (it only depends on  $q^2$ ). Provided we cut off the  $q$  integral in a way that preserves Lorentz invariance<sup>‡</sup> we can drop terms in the numerator that are odd in  $q$ . We can also replace  $q^\mu q^\nu \rightarrow \frac{1}{4}g^{\mu\nu}q^2$ . Shuffling terms a bit for reasons that will become clear later, we're left with an amplitude which we split up as

$$-i\mathcal{M} = -i\mathcal{M}^{(1)} - i\mathcal{M}^{(2)} \tag{C.2}$$

$$-i\mathcal{M}^{(1)} = 2e^2Q^2 g^{\mu\nu} \int_0^1 dx \int \frac{d^4q}{(2\pi)^4} \frac{q^2 + 2k^2x(1-x) - 2m^2}{(q^2 + k^2x(1-x) - m^2)^2} \tag{C.3}$$

$$-i\mathcal{M}^{(2)} = -4e^2Q^2 (g^{\mu\nu}k^2 - k^\mu k^\nu) \int_0^1 dx \int \frac{d^4q}{(2\pi)^4} \frac{2x(1-x)}{(q^2 + k^2x(1-x) - m^2)^2} \tag{C.4}$$

First let's study  $\mathcal{M}^{(1)}$ . Defining  $\tilde{m}^2 = m^2 - k^2x(1-x)$  we have the momentum integral

$$\begin{aligned} \int \frac{d^4q}{(2\pi)^4} \frac{q^2 - 2\tilde{m}^2}{(q^2 - \tilde{m}^2)^2} &= -i \int \frac{d^4q_E}{(2\pi)^4} \frac{q_E^2 + 2\tilde{m}^2}{(q_E^2 + \tilde{m}^2)^2} \\ &= -\frac{i}{8\pi^2} \int_0^\Lambda dq_E \frac{q_E^3 (q_E^2 + 2\tilde{m}^2)}{(q_E^2 + \tilde{m}^2)^2} \\ &= -\frac{i}{16\pi^2} \frac{\Lambda^4}{\Lambda^2 + \tilde{m}^2} \\ &= -\frac{i}{16\pi^2} (\Lambda^2 - \tilde{m}^2 + \mathcal{O}(1/\Lambda^2)) \end{aligned}$$

<sup>†</sup> Shifting variables of integration is legitimate for a convergent integral. For a divergent integral you need to have a cut-off in mind, say  $|p_E| < \Lambda$ , and you need to remember that the shift of integration variables changes the cutoff.

<sup>‡</sup> So really a better cutoff to have in mind would be  $|q_E| < \Lambda$ .

where we Wick rotated and introduced a momentum cutoff  $\Lambda$ . Thus

$$-i\mathcal{M}^{(1)} = -\frac{ie^2Q^2}{8\pi^2}g^{\mu\nu}\left(\Lambda^2 - m^2 + \frac{1}{6}k^2\right)$$

where we're neglecting terms that vanish as  $\Lambda \rightarrow \infty$ . In principle we can write down a low-energy effective action for the photon  $\Gamma[A]$  which incorporates the effects of the electron loop. Setting  $\Gamma[A] = \Gamma^{(1)} + \dots$  and matching to the amplitude  $\mathcal{M}^{(1)}$  fixes

$$\Gamma^{(1)} = -\int d^4x \frac{e^2Q^2}{8\pi^2}\left((\Lambda^2 - m^2)A_\mu A^\mu - \frac{1}{6}A_\mu\partial_\lambda\partial^\lambda A^\mu\right). \quad (\text{C.5})$$

We have a photon mass term plus a correction to the photon kinetic term. What's disturbing is that none of the terms in (C.5) are gauge invariant. This seems to contradict our claim in chapter 7 that a low energy effective action should respect all symmetries of the underlying theory.

The symmetry violation we found is due to the fact that we regulated the diagram with a momentum cutoff. This breaks gauge invariance and generates non-invariant terms in the effective action.<sup>†</sup> However the non-invariant terms we generated are *local*, meaning they are of the form  $\Gamma^{(1)} = \int d^4x \mathcal{L}^{(1)}(x)$ . (This is in contrast to the anomaly phenomenon discussed in chapter 13 where non-local terms arose.) With local violation a simple way to restore the symmetry is to modify the action of the underlying theory by subtracting off the induced symmetry-violating terms: that is, by changing the underlying QED Lagrangian  $\mathcal{L}_{QED} \rightarrow \mathcal{L}_{QED} - \mathcal{L}^{(1)}$ . To  $\mathcal{O}(e^2)$  in perturbation theory this modification exactly compensates for the non-gauge-invariance of the regulator and yields a gauge-invariant low-energy effective action. The procedure amounts to just dropping  $\mathcal{M}^{(1)}$  from the amplitude. (Another approach, developed in the homework, is to avoid generating the non-invariant terms in the first place by using a cutoff which respects the symmetry.)

Having argued that we can discard  $\mathcal{M}^{(1)}$ , let's return to the amplitude  $\mathcal{M}^{(2)}$  given in (C.4). Thanks to the prefactor  $g^{\mu\nu}k^2 - k^\mu k^\nu$  note that  $\mathcal{M}^{(2)}$  vanishes when dotted into  $k_\mu$ . This means  $\mathcal{M}^{(2)}$  corresponds to gauge-invariant terms in the effective action: terms which are invariant under  $A_\mu \rightarrow A_\mu + \partial_\mu\alpha$ , or equivalently under a shift of polarization  $\epsilon_\mu \rightarrow \epsilon_\mu + k_\mu$ .

<sup>†</sup> You can think of a momentum cutoff as a cutoff on the eigenvalues of the ordinary derivative  $\partial_\mu$ . Gauge invariance would require a cutoff on the eigenvalues of the covariant derivative  $\mathcal{D}_\mu = \partial_\mu + ieQA_\mu$ .

So  $\mathcal{M}^{(2)}$  gives our final result for the vacuum polarization, namely

$$-i\mathcal{M} = -4e^2 Q^2 (g^{\mu\nu} k^2 - k^\mu k^\nu) \int_0^1 dx \int_{|q_E| < \Lambda} \frac{d^4 q}{(2\pi)^4} \frac{2x(1-x)}{(q^2 + k^2 x(1-x) - m^2)^2} \quad (\text{C.6})$$

where  $\Lambda$  is a momentum cutoff. The integrals can be evaluated but lead to rather complicated expressions. For most purposes it's best to leave the result in the form (C.6).

## References

REGULATORS AND SYMMETRIES. For more discussion of the connection between regulators and symmetries of the effective action see section 13.1.3.

GAUGE INVARIANT CUTOFFS. Many gauge-invariant regulators have been developed. One such scheme, Pauli-Villars regularization, is described in problem C.1. It's also discussed by A. Zee, *Quantum field theory in a nutshell* on p. 151 and applied to vacuum polarization in chapter III.7. Another gauge-invariant scheme, dimensional regularization, is described in Peskin & Schroeder section 7.5.

DECOUPLING. Decoupling of heavy particles at low energies is discussed in Donoghue et. al. section VI-2.

SCHEME DEPENDENCE. The scheme dependence of running couplings, and the fact that  $\beta$ -functions are scheme independent to two loops, is discussed in Weinberg vol. II p. 138.

## Exercises

### C.1 Pauli-Villars regularization

A gauge-invariant scheme for cutting off loop integrals is to subtract the contribution of heavy Pauli-Villars regulator fields. These are fictitious particles whose masses are chosen to make loop integrals converge. Denoting the vacuum polarization amplitude (C.2) by  $-i\mathcal{M}(m)$  we define the Pauli-Villars regulated amplitude by

$$-i\mathcal{M} = \sum_{i=0}^3 -ia_i \mathcal{M}(m_i).$$

Here  $a_i = (1, -1, -1, 1)$  and  $m_i = (m, M, M, \sqrt{2M^2 - m^2})$  have been chosen so that

$$\sum_i a_i = \sum_i a_i m_i^2 = 0.$$

One says we've introduced three Pauli-Villars regulator fields. The idea is that for fixed  $\Lambda$  we can send  $M \rightarrow \infty$  and recover our original amplitude. However for fixed  $M$  we can send  $\Lambda \rightarrow \infty$ ; in this limit the regulator mass  $M$  serves to cut off the loop integral in a gauge-invariant way.

- (i) Consider the term  $\mathcal{M}^{(1)}$  in the amplitude. After summing over regulators show that for fixed  $M$  you can send the momentum cut-off  $\Lambda \rightarrow \infty$ , and show that in this limit  $\mathcal{M}^{(1)}$  vanishes identically. This reflects the fact that the effective action is gauge invariant when you use a gauge-invariant cutoff.
- (ii) The term  $\mathcal{M}^{(2)}$  in the amplitude is only logarithmically divergent and can be made finite by subtracting the contribution of a single regulator field. So in the Pauli-Villars scheme our final expression for the vacuum polarization is

$$-i\mathcal{M} = -4e^2 Q^2 (g^{\mu\nu} k^2 - k^\mu k^\nu) \int_0^1 dx \int \frac{d^4 q}{(2\pi)^4} \left[ \frac{2x(1-x)}{(q^2 + k^2 x(1-x) - m^2)^2} - (m^2 \rightarrow M^2) \right].$$

Use this result to find the running coupling  $e^2(M)$  in the Pauli-Villars scheme. You could do this, for example, by redoing problem 7.4 parts (ii) and (iii) with a Pauli-Villars cutoff.

## C.2 Mass-dependent renormalization

In problem C.1 you made a mass-independent subtraction to regulate the loop integral, replacing (for  $k = 0$ )

$$\frac{1}{(q^2 - m^2)^2} \rightarrow \frac{1}{(q^2 - m^2)^2} - \frac{1}{(q^2 - M^2)^2}.$$

Provided  $M \gg m$  this serves to cut off the loop integral at  $q^2 \approx M^2$ . However if one is interested in the behavior of the coupling at energies which are small compared to the electron mass it's more physical to make a mass-dependent subtraction and replace

$$\frac{1}{(q^2 - m^2)^2} \rightarrow \frac{1}{(q^2 - m^2)^2} - \frac{1}{(q^2 - m^2 - M^2)^2}.$$

This subtraction serves as a good cut-off even for small  $M$  (note that it makes the loop integral vanish as  $M \rightarrow 0$ ).

- (i) Find the running coupling  $e^2(M)$  with this new cutoff. It's convenient to set the renormalization scale  $\mu$  to zero, that is, to solve for  $e^2(M)$  in terms of  $e^2(0)$ .
- (ii) Expand your answer to find how  $e^2(M)$  behaves for  $M \gg m$  and for  $M \ll m$ . Make a qualitative sketch of  $e^2(M)$ .

Moral of the story: the running couplings of problems C.1 and C.2 are said to be evaluated in different renormalization schemes. Yet another scheme is the momentum cutoff used in chapter 7. The choice of scheme is up to you; physical quantities if calculated exactly are the same in every scheme. Mass-independent schemes are often easier to work with. But mass-dependent schemes have certain advantages, in particular they incorporate “decoupling” (the fact that heavy particles drop out of low-energy dynamics). Also note that at high energies the running couplings are independent of mass, and are the same whether computed with a momentum cutoff or a Pauli-Villars cutoff. This reflects a general phenomenon, discussed in the references: at high energies the first two terms in the perturbative expansion of a  $\beta$ -function are independent of scheme.



# Appendix D

## Two-component spinors

Physics 85200

January 6, 2014

For the most part we've described fermions in terms of four-component Dirac spinors. This is very convenient for QED. However it becomes awkward when discussing chiral theories, or theories that violate fermion number, since one is forced to use lots of chiral projection and charge conjugation operators. In this appendix we introduce a more general and flexible notation for fermions: two-component chiral spinors.

As discussed in section 4.1 a Dirac spinor  $\psi_D$  can be decomposed into

$$\psi_D = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$$

where  $\psi_L$  and  $\psi_R$  are two-component chiral spinors, left- and right-handed respectively. Under a Lorentz transformation

$$\psi_L \rightarrow e^{-i(\vec{\theta} - i\vec{\phi}) \cdot \vec{\sigma}/2} \psi_L \quad \psi_R \rightarrow e^{-i(\vec{\theta} + i\vec{\phi}) \cdot \vec{\sigma}/2} \psi_R. \quad (\text{D.1})$$

So left- and right-handed spinors don't mix under Lorentz transformations: they're irreducible representations of the Lorentz group.

The Dirac Lagrangian can be expressed in two-component notation as

$$\begin{aligned} \mathcal{L} &= \bar{\psi}_D i \gamma^\mu \partial_\mu \psi_D - m \bar{\psi}_D \psi_D \\ &= i \psi_L^\dagger \bar{\sigma}^\mu \partial_\mu \psi_L + i \psi_R^\dagger \sigma^\mu \partial_\mu \psi_R - m \left( \psi_L^\dagger \psi_R + \psi_R^\dagger \psi_L \right). \end{aligned} \quad (\text{D.2})$$

Here we've defined the  $2 \times 2$  analogs of the Dirac matrices

$$\sigma^\mu = (\mathbf{1}; \vec{\sigma}) \quad \bar{\sigma}^\mu = (\mathbf{1}; -\vec{\sigma})$$

(the overbar on  $\bar{\sigma}$  is just part of the name – it doesn't indicate complex conjugation). As pointed out in section 4.1, in the massless limit the left-

and right-handed parts of a Dirac spinor are decoupled and behave as independent fields.

It turns out that complex conjugation interchanges left- and right-handed spinors. More precisely, as you'll show on the homework,

$$\begin{aligned} \epsilon\psi_L^* & \text{ is a right-handed spinor} \\ -\epsilon\psi_R^* & \text{ is a left-handed spinor} \end{aligned} \quad (\text{D.3})$$

Here  $\epsilon = i\sigma^2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$  is a  $2 \times 2$  antisymmetric matrix. This sort of relation means that any theory can be expressed purely in terms of left-handed (or right-handed) spinors.<sup>†</sup> That is, for a given physical theory, the choice of spinor chirality is just a matter of convention.<sup>‡</sup>

The advantage of working with chiral spinors is that it's easy to generalize (D.2). For instance we can write down a theory of a single massive chiral fermion.

$$\mathcal{L} = i\psi_L^\dagger \bar{\sigma}^\mu \partial_\mu \psi_L + \frac{1}{2}m \left( \psi_L^T \epsilon \psi_L - \psi_L^\dagger \epsilon \psi_L^* \right) \quad (\text{D.4})$$

Here we're using the fact that  $\psi_L^T \epsilon \psi_L$  is Lorentz invariant, and we've added the complex conjugate  $-\psi_L^\dagger \epsilon \psi_L^*$  to keep the Lagrangian real. More generally, with  $N$  chiral fermions  $\psi_{Li}$  we could have

$$\mathcal{L} = i\psi_{Li}^\dagger \bar{\sigma}^\mu \partial_\mu \psi_{Li} + \frac{1}{2}m_{ij}\psi_{Li}^T \epsilon \psi_{Lj} - \frac{1}{2}m_{ij}^*\psi_{Li}^\dagger \epsilon \psi_{Lj}^* \quad (\text{D.5})$$

We've seen that mass term before: it's the Majorana mass term for neutrinos (14.6). By Fermi statistics the mass matrix  $m_{ij}$  can be taken to be symmetric. Note that the kinetic terms have a  $U(N)$  symmetry  $\psi_{Li} \rightarrow U_{ij}\psi_{Lj}$  which in general is broken by the mass term.

Although chiral spinors make it easy to write the most general fermion Lagrangian, one can always revert to Dirac notation. To pick out the left- and right-handed pieces of a Dirac spinor one uses projection operators.

$$\begin{pmatrix} \psi_L \\ 0 \end{pmatrix} = \frac{1 - \gamma^5}{2} \psi_D \quad \begin{pmatrix} 0 \\ \psi_R \end{pmatrix} = \frac{1 + \gamma^5}{2} \psi_D$$

And to capture complex conjugation – for instance the  $\epsilon\psi_L^*$  appearing in (D.4) – one uses charge conjugation. Recall that the charge conjugate of a

<sup>†</sup> For instance you could rewrite the Dirac Lagrangian in terms of two left-handed spinors, namely  $\psi_1 = \psi_L$  and  $\psi_2 = -\epsilon\psi_R^*$ .

<sup>‡</sup> Another matter of convention: chiral spinors can be re-expressed using the Majorana spinors described in the homework.

Dirac spinor is defined by

$$\psi_{DC} = -i\gamma^2\psi_D^* = \begin{pmatrix} -\epsilon\psi_R^* \\ \epsilon\psi_L^* \end{pmatrix}.$$

The fact that  $\psi_{DC}$  really is a Dirac spinor is a restatement of (D.3).

### References

We've described two-component spinors using matrix notation. It's more common to introduce a specialized index notation. See Wess & Bagger, *Supersymmetry and supergravity*, appendix A.

### Exercises

#### D.1 Chirality and complex conjugation

Show that complex conjugation changes the chirality of a spinor. That is, show that the behavior under Lorentz transformations (D.3) follows directly from (D.1). It helps to note that  $\vec{\sigma}^* = \epsilon\vec{\sigma}\epsilon$ .

#### D.2 Lorentz invariant bilinears

Show that the fermion bilinears  $\psi_L^T\epsilon\psi_L$  and  $\psi_R^T\epsilon\psi_R$  are invariant under Lorentz transformations. It helps to note that  $\vec{\sigma}^T = \epsilon\vec{\sigma}\epsilon$ .

#### D.3 Majorana spinors

- (i) A Majorana spinor  $\psi_M$  is a Dirac spinor that satisfies  $\psi_{MC} = \psi_M$ . Given a two-component left-handed spinor  $\psi_L$ , show that one can build a Majorana spinor by setting

$$\psi_M = \begin{pmatrix} \psi_L \\ \epsilon\psi_L^* \end{pmatrix}.$$

- (ii) The Lagrangian for a free Majorana spinor of mass  $m$  is

$$\mathcal{L} = \frac{1}{2}\bar{\psi}_M i\gamma^\mu \partial_\mu \psi_M - \frac{1}{2}m\bar{\psi}_M \psi_M.$$

Rewrite this Lagrangian in terms of  $\psi_L$ . Do you reproduce (D.4)?

**D.4 Dirac spinors**

Consider two left-handed spinors  $\psi_{L1}$ ,  $\psi_{L2}$  described by the Lagrangian (D.5).

- (i) Impose a  $U(1)$  symmetry  $\psi_{L1} \rightarrow e^{i\theta}\psi_{L1}$ ,  $\psi_{L2} \rightarrow e^{-i\theta}\psi_{L2}$  on the Lagrangian.
- (ii) Show that the resulting theory is equivalent to the Dirac Lagrangian (D.2).
- (iii) What is the interpretation of the  $U(1)$  symmetry in Dirac language?

# Appendix E

## Summary of the standard model

Physics 85200

January 6, 2014

For background see chapter 12. Here I'll just go through the main results.

Gauge structure:

The gauge group is  $SU(3)_C \times SU(2)_L \times U(1)_Y$  with gauge fields  $G_\mu$ ,  $W_\mu$ ,  $B_\mu$  and gauge couplings  $g_s$ ,  $g$ ,  $g'$ . The field strengths are denoted  $G_{\mu\nu}$ ,  $W_{\mu\nu}$ ,  $B_{\mu\nu}$ . In place of  $g$  and  $g'$  we'll often work in terms of the electromagnetic coupling  $e$ , the  $Z$  coupling  $g_Z$ , and the weak mixing angle  $\theta_W$ , defined by

$$\begin{aligned}e &= gg' / \sqrt{g^2 + g'^2} \\g_Z &= \sqrt{g^2 + g'^2} \\\cos \theta_W &= g / \sqrt{g^2 + g'^2} \\\sin \theta_W &= g' / \sqrt{g^2 + g'^2}\end{aligned}$$

At the scale  $m_Z$  the values are

$$\begin{aligned}\alpha_s &= g_s^2 / 4\pi = 0.119 \\\alpha &= e^2 / 4\pi = 1/128 \quad (\text{vs. } 1/137 \text{ at low energies}) \\\alpha_Z &= \frac{(g_Z/2)^2}{4\pi} = 1/91 \\\sin^2 \theta_W &= 0.231\end{aligned}$$

Another useful combination is the Fermi constant, related to the  $W$  mass  $m_W$  and the Higgs vev  $v$  by

$$G_F = \frac{g^2}{4\sqrt{2}m_W^2} = \frac{1}{\sqrt{2}v^2} = 1.17 \times 10^{-5} \text{ GeV}^{-2}.$$

Matter content:

<i>field</i>	<i>gauge quantum numbers</i>
left-handed leptons $L_i = \begin{pmatrix} \nu_{Li} \\ e_{Li} \end{pmatrix}$	$(\mathbf{1}, \mathbf{2}, -1)$
right-handed leptons $e_{Ri}$	$(\mathbf{1}, \mathbf{1}, -2)$
left-handed quarks $Q_i = \begin{pmatrix} u_{Li} \\ d_{Li} \end{pmatrix}$	$(\mathbf{3}, \mathbf{2}, 1/3)$
right-handed up-type quarks $u_{Ri}$	$(\mathbf{3}, \mathbf{1}, 4/3)$
right-handed down-type quarks $d_{Ri}$	$(\mathbf{3}, \mathbf{1}, -2/3)$
Higgs doublet $\phi$	$(\mathbf{1}, \mathbf{2}, 1)$
conjugate Higgs doublet $\tilde{\phi} = \epsilon\phi^*$	$(\mathbf{1}, \mathbf{2}, -1)$

Here  $i$  is a three-valued generation index and  $\epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$  is an  $SU(2)$ -invariant tensor. The fermions are all chiral spinors, either left- or right-handed; I'll write them as 4-component Dirac fields although only two of the components are non-zero.

Lagrangian:

The most general renormalizable gauge-invariant Lagrangian is

$$\mathcal{L} = \mathcal{L}_{\text{Dirac}} + \mathcal{L}_{\text{Yang-Mills}} + \mathcal{L}_{\text{Higgs}} + \mathcal{L}_{\text{Yukawa}}$$

$$\mathcal{L}_{\text{Dirac}} = \bar{L}_i i\gamma^\mu \mathcal{D}_\mu L_i + \bar{e}_{Ri} i\gamma^\mu \mathcal{D}_\mu e_{Ri} + \bar{Q}_i i\gamma^\mu \mathcal{D}_\mu Q_i + \bar{u}_{Ri} i\gamma^\mu \mathcal{D}_\mu u_{Ri} + \bar{d}_{Ri} i\gamma^\mu \mathcal{D}_\mu d_{Ri}$$

$$\mathcal{L}_{\text{Yang-Mills}} = -\frac{1}{2} \text{Tr} (G_{\mu\nu} G^{\mu\nu}) - \frac{1}{2} \text{Tr} (W_{\mu\nu} W^{\mu\nu}) - \frac{1}{4} B_{\mu\nu} B^{\mu\nu}$$

$$\mathcal{L}_{\text{Higgs}} = \mathcal{D}_\mu \phi^\dagger \mathcal{D}^\mu \phi + \mu^2 \phi^\dagger \phi - \lambda (\phi^\dagger \phi)^2$$

$$\mathcal{L}_{\text{Yukawa}} = -\Lambda_{ij}^e \bar{L}_i \phi e_{Rj} - \Lambda_{ij}^d \bar{Q}_i \phi d_{Rj} - \Lambda_{ij}^u \bar{Q}_i \tilde{\phi} u_{Rj} + \text{c.c.}$$

The covariant derivative is  $\mathcal{D}_\mu = \partial_\mu + ig_s G_\mu + ig W_\mu + \frac{ig'}{2} B_\mu Y$ , where the gauge fields are taken to act in the appropriate representation (for example  $G_\mu Q_i = \frac{1}{2} G_\mu^a \lambda^a Q_i$  where  $\lambda^a$  are the Gell-Mann matrices, while  $G_\mu L_i = 0$  since  $L_i$  is a color singlet. Likewise  $W_\mu Q_i = \frac{1}{2} W_\mu^a \sigma^a Q_i$  but  $W_\mu e_{Ri} = 0$ .)

Conventions:

We assume  $\mu^2 > 0$  so the gauge symmetry is spontaneously broken. The standard gauge choice is to set

$$\phi = \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}}(v + H) \end{pmatrix} \quad \tilde{\phi} = \begin{pmatrix} \frac{1}{\sqrt{2}}(v + H) \\ 0 \end{pmatrix}$$

where  $v = \mu/\sqrt{\lambda} = 246 \text{ GeV}$  is the electroweak vev and  $H$  is the physical (real scalar) Higgs field. One conventionally redefines the fermions to diagonalize the Yukawa couplings at the price of getting a unitary CKM mixing matrix  $V_{ij}$  in the quark – quark –  $W^\pm$  couplings; see section 12.3 for details. At this point we'll switch notation and assemble the left- and right-handed parts of the various fermions into 4-component mass eigenstate Dirac spinors.

Mass spectrum:

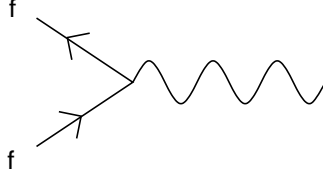
$$\begin{aligned} m_W^2 &= \frac{1}{4}g^2v^2 \\ m_Z^2 &= \frac{1}{4}g_Z^2v^2 = m_W^2/\cos^2\theta_W \\ m_H^2 &= 2\mu^2 \\ m_f &= \frac{\lambda_f v}{\sqrt{2}} \quad f = \text{any fermion} \end{aligned}$$

Here  $\lambda_f$  is the Yukawa coupling for  $f$  (after diagonalizing). The observed masses are

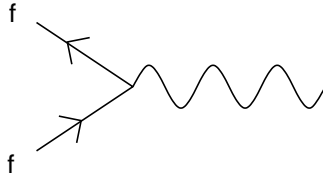
$m_W = 80.4 \text{ GeV}$		$m_Z = 91.2 \text{ GeV}$
$m_e = 0.511 \text{ MeV}$	$m_\mu = 106 \text{ MeV}$	$m_\tau = 1780 \text{ MeV}$
$m_u = 3 \text{ MeV}$	$m_c = 1.3 \text{ GeV}$	$m_t = 172 \text{ GeV}$
$m_d = 5 \text{ MeV}$	$m_s = 100 \text{ MeV}$	$m_b = 4.2 \text{ GeV}$

Vertex factors:

The vertices arising from  $\mathcal{L}_{\text{Dirac}}$  are

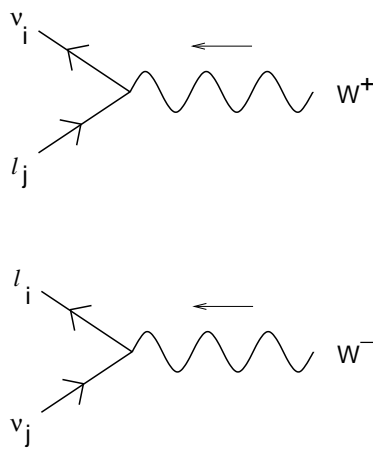


$$\text{f} \quad \text{f} \quad \text{A} \quad -ieQ\gamma^\mu \quad Q = \text{electric charge}$$

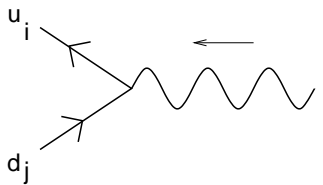


$$\text{f} \quad \text{f} \quad \text{Z} \quad -\frac{ig_Z}{2} (c_V \gamma^\mu - c_A \gamma^\mu \gamma^5) \quad c_V = T_L^3 - 2 \sin^2 \theta_W Q, \quad c_A = T_L^3$$

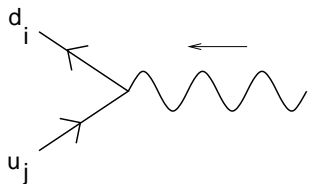
$$\text{where } T_L^3 = \begin{cases} 1/2 & \text{for neutrinos and up-type quarks} \\ -1/2 & \text{for charged leptons and down-type quarks} \end{cases}$$



$$\left. \begin{array}{l} \text{v}_i \quad \text{l}_j \quad \text{W}^+ \\ \text{l}_i \quad \text{v}_j \quad \text{W}^- \end{array} \right\} \text{both } -\frac{ig}{2\sqrt{2}} \gamma^\mu (1 - \gamma^5) \delta_{ij}$$

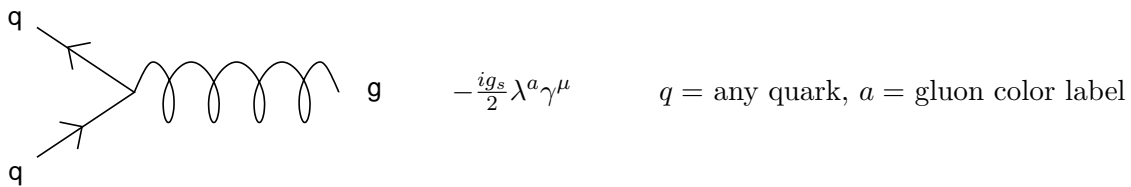


$$\text{u}_i \quad \text{d}_j \quad \text{W}^+ \quad -\frac{ig}{2\sqrt{2}} \gamma^\mu (1 - \gamma^5) V_{ij} \quad \text{where } V_{ij} \text{ is the CKM matrix}$$

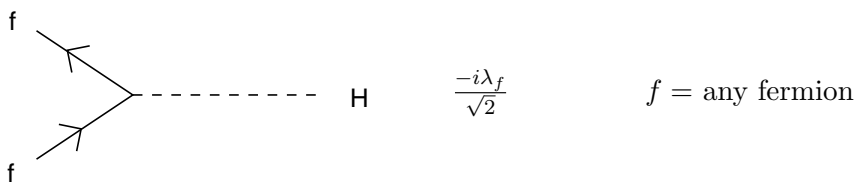


$$\text{d}_i \quad \text{u}_j \quad \text{W}^- \quad -\frac{ig}{2\sqrt{2}} \gamma^\mu (1 - \gamma^5) (V^\dagger)_{ij}$$

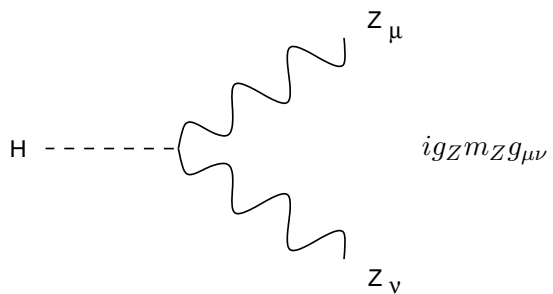
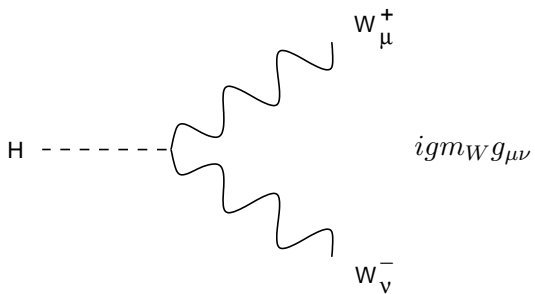
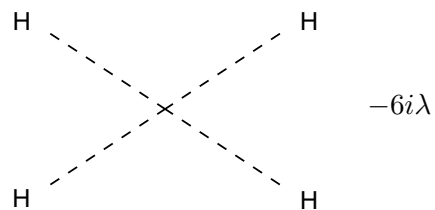
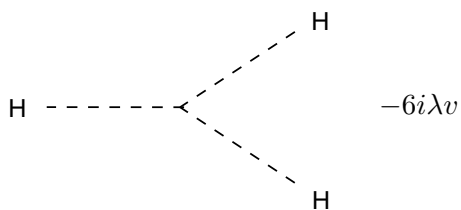


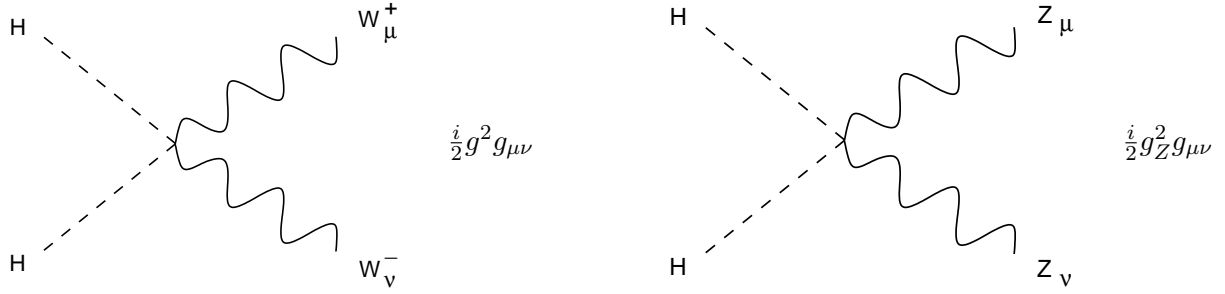


The vertex arising from  $\mathcal{L}_{\text{Yukawa}}$  is

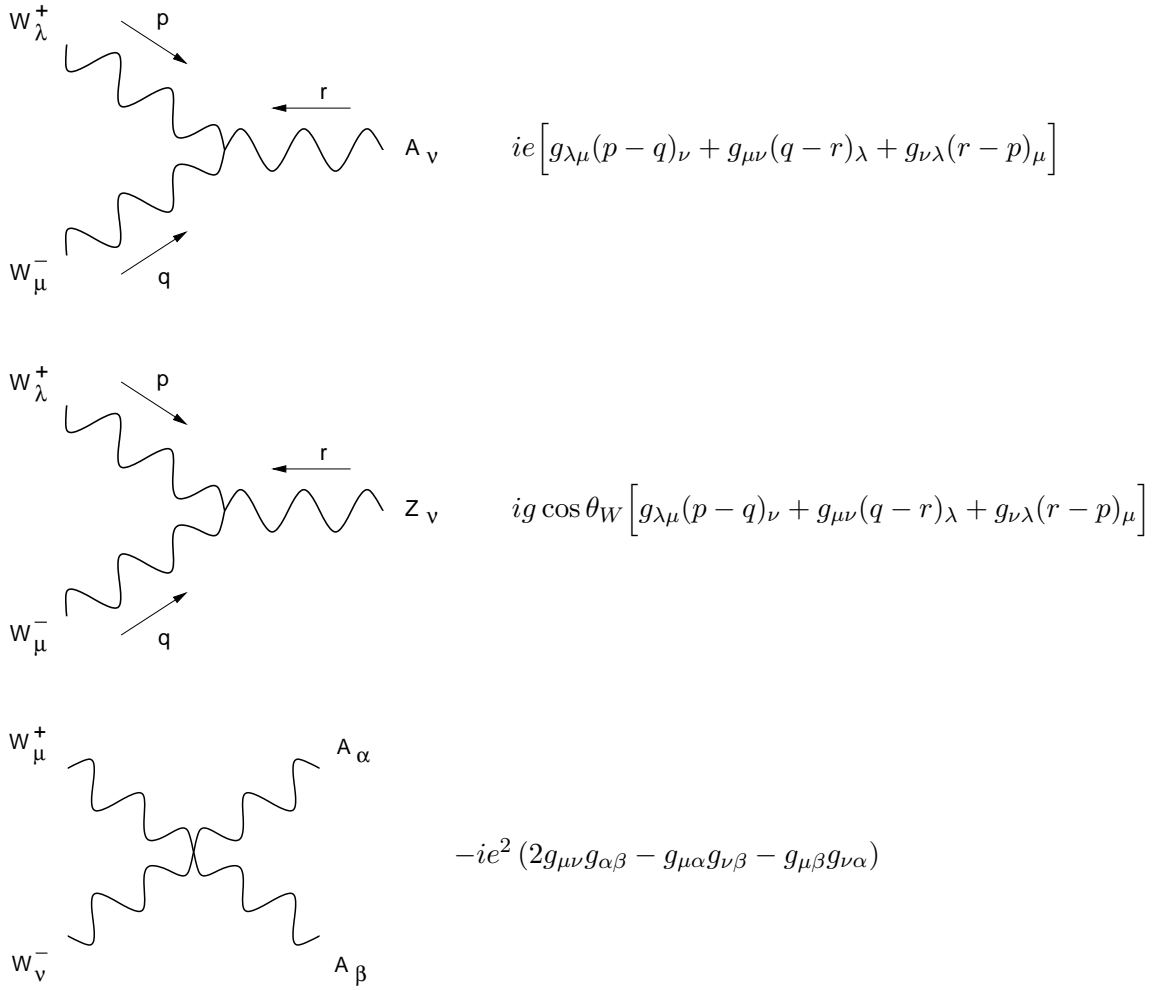


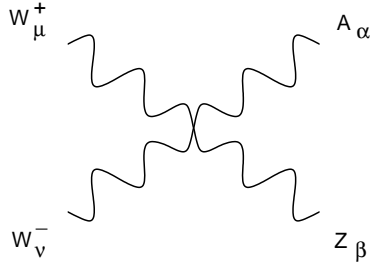
The vertices arising from  $\mathcal{L}_{\text{Higgs}}$  are



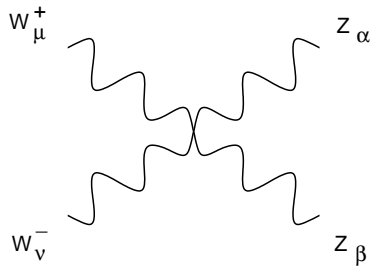


The vertices arising from  $\mathcal{L}_{\text{Yang-Mills}}$  are

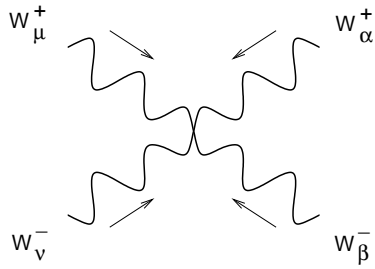




$$-ieg \cos \theta_W (2g_{\mu\nu}g_{\alpha\beta} - g_{\mu\alpha}g_{\nu\beta} - g_{\mu\beta}g_{\nu\alpha})$$



$$-ig^2 \cos^2 \theta_W (2g_{\mu\nu}g_{\alpha\beta} - g_{\mu\alpha}g_{\nu\beta} - g_{\mu\beta}g_{\nu\alpha})$$



$$ig^2 (2g_{\mu\alpha}g_{\nu\beta} - g_{\mu\beta}g_{\nu\alpha} - g_{\mu\nu}g_{\alpha\beta})$$

(note all particles directed inwards)

The additional vertices from  $\mathcal{L}_{\text{Yang-Mills}}$  describing gluon 3- and 4-point couplings are given in chapter 10.