Magnetism, Radiation, and Relativity

Supplementary notes for a calculus-based
introductory physics course

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Preface

In the early 1960s, Edward M. Purcell wrote an innovative electromagnetism text (*Electricity and Magnetism*: *Berkeley Physics Course Volume 2*, published by McGraw-Hill, now in its second edition) in which he used relativistic arguments to derive the existence of magnetism and radiation. This approach to the physics of moving charges brings the subject to life, illustrating the physical origin of many important electrodynamic phenomena in a pictorial way. Furthermore, this approach is mathematically easier than the more traditional approach based on the Biot-Savart law, Ampere's law, and Maxwell's equations.

Unfortunately, Purcell's textbook is written at a level that is too advanced for the typical introductory physics course. In these notes I have therefore tried to extract the essential relativistic arguments of Purcell's approach, simplify them somewhat, and present them in a format that could be incorporated into any calculus-based introductory course taught out of a more typical textbook. Each of the five "lessons" can be covered in about one class session. Students should already be familiar with electrostatic fields and Gauss's law, basic magnetic phenomena such as forces between parallel currents, and the rudiments of relativity (including reference frames, length contraction, and the speed of light as a limit for transmitting information). No prior knowledge of magnetic fields, electrodynamics, or more advanced notions of relativity (Lorentz transformation, relativistic dynamics) is required.

Although nearly every idea in these notes is borrowed from Purcell's book, the illustrations have all been redrawn and the text is entirely my own. Instructors have my permission to duplicate these notes for classroom use as needed. Comments and suggestions from both instructors and students would be most welcome.

Daniel V. Schroeder
2 January 1999

Lesson 1

Transformation of the Electric Field

If you have an electric field in one reference frame, how does it look from a different reference frame, moving with respect to the first? This question is crucial if we are to understand fields created by moving sources. For now, let us restrict ourselves to the special case where the sources that create the field are at rest with respect to one of the reference frames. So our question is: Given the electric field in the frame where the sources are at rest, what is the electric field in some other frame?

Our fundamental assumption will be that knowing the electric field at some point (in space and time) in the rest frame of the sources, and knowing the relative velocity of the two frames, gives us all the information we need to calculate the electric field at the same point in the other frame. In other words, the electric field in the other frame does *not* depend on the particular distribution of the source charges, only on the *local* value of the electric field in the first frame at that point. Basically, we're taking the electric field very seriously, assuming that it is a complete representation of the influence of the far-away charges.

1.1 A Uniform Electric Field

Let us begin with a very simple situation: a charged parallel-plate capacitor, whose electric field (in its rest frame) is uniform between the plates and zero outside (neglecting edge effects). What is the electric field of this charge distribution in a reference frame where it is in motion?

Suppose first that the motion is in a direction parallel to the plates (see figure 1.1). The plates are then shorter by a factor of $\sqrt{1-(v/c)^2}$ than they are in their rest frame, but the distance between them is the same. The total charge on each plate is also the same, since charge is a frame-independent quantity. The charge *per unit area* on the plates is therefore larger than in the rest frame by a factor of $1/\sqrt{1-(v/c)^2}$, and we will therefore find that the field between them is stronger by this factor.

Let's back up and derive this carefully. First consider the electric field of a single, infinite plate of positive charge, moving parallel to itself. What can we say about the value of this field? It must be uniform both above and below the plate, since it is uniform in its rest frame, and we are assuming that knowing the field in one frame is enough to know it in the other. But we cannot (yet) rule out the possibility that the field of the plate *could* have

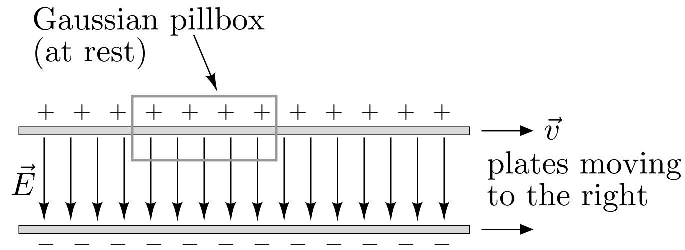

Figure 1.1. Two oppositely charged parallel plates produce a uniform electric field, even when they are moving as shown. The Gaussian pillbox can be used to find the strength of the field.

a nonzero component along the direction of motion, as shown in figure 1.2a. However, even if this were the case, the field of an infinite plate of negative charge would have to be equal and opposite to the field of the positive plate (as shown in figure 1.2b), since combining the plates must cancel the fields exactly (neutral objects produce no fields). If we separate the plates of charge, the horizontal components of their fields still cancel exactly, and we are left with a uniform vertical field between the plates and zero field outside, as shown in figure 1.1.

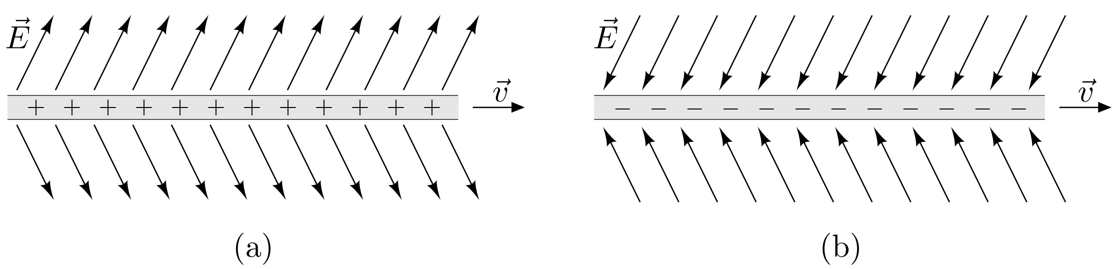

Figure 1.2. An infinite sheet of positive charge, moving to the right, could conceivably create a field like that shown in (a). But even if it did, a sheet of negative charge would have an opposite field, as in (b).

Now imagine a Gaussian pillbox that straddles one of the plates, with one face outside and one face inside the region of nonzero electric field (see figure 1.1). Applying Gauss's law to this pillbox, you can now show that the magnitude of the electric field between the plates is

$$|\vec{E}'| = \frac{\sigma'}{\epsilon_0}, \tag{1.1}$$

where $'$ denotes a value measured in the frame where the plates are moving, and σ is the surface charge density of the positive plate. Since the plates are length-contracted by a factor of $\sqrt{1-(v/c)^2}$, the surface charge density in the primed frame is related to its value in the rest frame of the plates by

$$\sigma' = \frac{\sigma}{\sqrt{1-(v/c)^2}}. \tag{1.2}$$

But the electric field in the rest frame has magnitude σ/ϵ_0, and the field points in the same direction in both frames, so we can conclude that

$$\vec{E}' = \frac{\vec{E}}{\sqrt{1-(v/c)^2}} \qquad \text{(motion } \perp \text{ to } \vec{E}\text{)}. \tag{1.3}$$

This is for motion in a direction parallel to the plates, that is, perpendicular to the direction of $\vec{E}$. Notice that the electric field in the primed frame is *stronger* than in the unprimed frame.

What happens if the motion is in the direction perpendicular to the plates, that is, parallel to $\vec{E}$? In this case the length contraction does not affect the size of the plates, though it does reduce the distance between them. But the distance between a pair of closely spaced, uniformly charged plates does *not* affect the strength of the field between them. That is, for motion in a direction parallel to $\vec{E}$,

$$\vec{E}' = \vec{E} \qquad \text{(motion } \parallel \text{ to } \vec{E}\text{)}. \tag{1.4}$$

Finally, consider the most general case where the motion is in some diagonal direction relative to the field. In this case, we can consider the field to be a superposition of a field in the parallel direction and a field in the perpendicular direction, each generated by its own set of appropriately oriented plates, as shown in figure 1.3. (Remember that we assume that the precise nature of the source of the field is irrelevant.) The two sets of plates are then length-contracted as described above, and the two components of $\vec{E}$ are affected accordingly:

$$E'_\perp = \frac{E_\perp}{\sqrt{1-(v/c)^2}}; \qquad E'_\parallel = E_\parallel. \tag{1.5}$$

(Here $E_\perp$ refers to the components of $\vec{E}$ perpendicular to the motion, and $E_\parallel$ is the component of $\vec{E}$ parallel to the motion.) These are our final equations for the transformation of the electric field between reference frames. It is important to remember, however, that they apply only if the source of the field is at rest in the unprimed frame. Since there is always *some* reference frame in which any particular source is at rest, these equations are sufficient for solving a wide variety of problems.

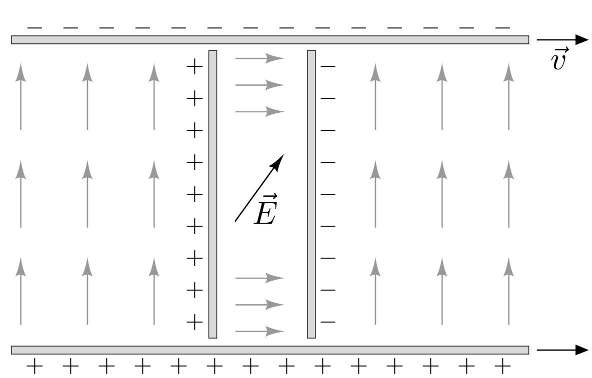

Figure 1.3. If the electric field points in a diagonal direction relative to the motion, it can be considered a superposition of a parallel field and a perpendicular field, each generated by a set of appropriately oriented plates.

Notice that the transformation law for the electric field vector is quite different from the transformation law for ordinary *displacement* vectors (which are contracted in the direction along the motion and unchanged in the perpendicular directions). In particular, it would be wrong to think of the electric field vector as a physical object that gets length-contracted just like any other object. In fact, the field vector is *longer*, not shorter, in the primed frame. Furthermore, the stretching occurs in the direction perpendicular to the motion, while the parallel component is unchanged.

Exercise 1.1. Imagine that in a certain region of space there is a uniform electric field whose magnitude, in the earth's reference frame, is 100 volts/meter, and whose direction, in the earth's reference frame, is inclined 45° with respect to the x axis. Find the magnitude and direction of the field in a reference frame moving in the x direction at 9/10 the speed of light with respect to the earth.

Equations (1.5), by the way, are sufficient to prove that the electric field of a single plane of charge does not look like figure 1.2; it must point perpendicular to the plane, even in a reference frame where the plane is moving.

1.2 The Field of a Moving Point Charge

The most important application of equations (1.5) is to the field of a single point charge, moving with constant velocity. In its *rest* frame the electric field of a positive point charge has the same strength in all directions and points directly away from the charge. What does this field look like in some other reference frame?

In applying equations (1.5) to a nonuniform electric field we have to be very careful, since we must keep track not only of *what* the value of the field is, but also *where* it has this value. Let us therefore imagine that our point charge is surrounded by a spherical shell—spherical, that is, in the rest frame of the particle, where it is also at rest. In our reference frame, however, both the particle and its sphere are moving. Length contraction therefore says that the sphere is flattened into a spheroid, as shown in cross-section in figure 1.4.

Now consider the value of the electric field at any point on the surface of the sphere. Let x and y be the components of the displacement, in the rest frame of the charge, from the charge to this point, measured parallel and perpendicular to the direction of motion, as shown in the figure. Since the field in the rest frame of the charge points directly away from the charge, its components are in the same ratio as the components of the displacement:

$$\frac{E_y}{E_x} = \frac{y}{x}. \tag{1.6}$$

In our reference frame, where the charge is moving, the displacement x' in the direction of motion is length-contracted:

$$x' = \sqrt{1-(v/c)^2}\,x \tag{1.7}$$

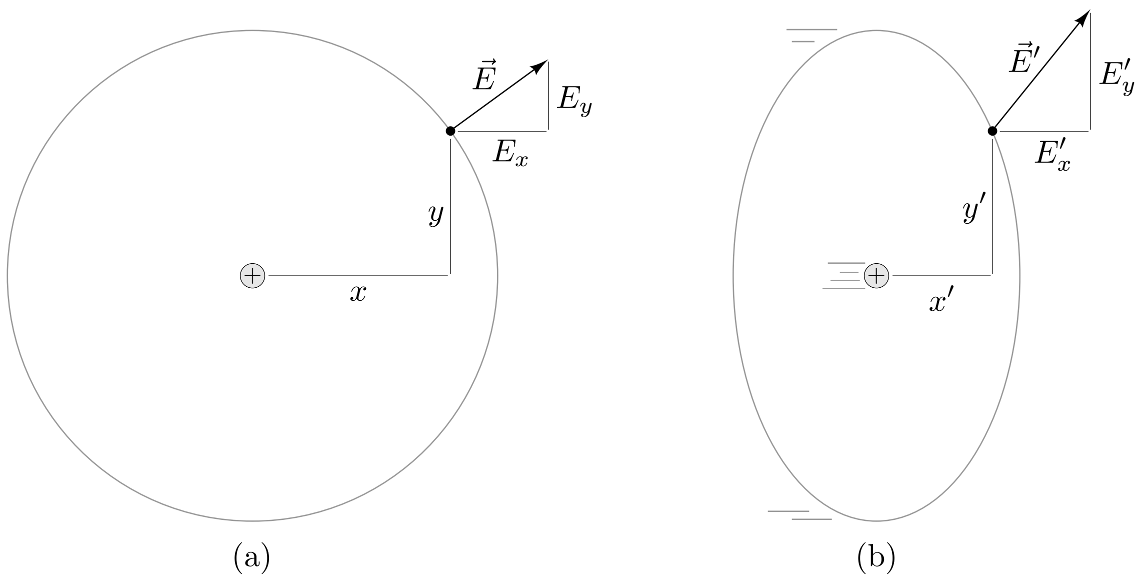

Figure 1.4. (a) A point charge at rest, surrounded by an imaginary sphere. The electric field at any point on the sphere points directly away from the charge. (b) In a reference frame where the charge and the sphere are moving to the right, the sphere is length-contracted but the vertical component of the field is stronger. These two effects combine to make the field again point directly away from the current location of the charge.

(while the y component of the displacement is the same in both frames). However, according to the results of the previous section, the y component of the field is enhanced by a similar factor:

$$E'_y = \frac{E_y}{\sqrt{1-(v/c)^2}} \tag{1.8}$$

(while the x component of the field is the same in both frames). The ratio of the field components is therefore

$$\frac{E'_y}{E'_x} = \frac{E_y}{E_x\sqrt{1-(v/c)^2}} = \frac{y}{x\sqrt{1-(v/c)^2}} = \frac{y'}{x'}. \tag{1.9}$$

In other words, the field in the primed frame points directly away from the charge, just as in the unprimed frame.

A sketch of the electric field of a point charge moving at constant velocity is shown in figure 1.5. The faster the charge is moving, the more noticeable the enhancement of the perpendicular component of the field becomes. If the speed of the charge is much less than the speed of light, this enhancement is often negligible. But in the following lesson we'll see that under certain circumstances, it is crucially important even at low velocities.

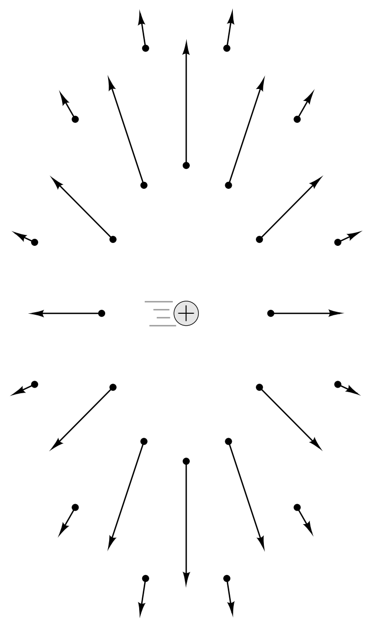

Figure 1.5. The electric field of a point charge moving to the right with a constant velocity, 4/5 the speed of light.

Lesson 2

Magnetic Forces

I experienced a miracle . . . as a child of four or five when my father showed me a compass.

—Albert Einstein

We are now ready to understand the origin of magnetic forces. I assume that you are already familiar with the basic *phenomena* of magnetism: permanent magnets with "poles" that either attract or repel; magnetic forces exerted on moving charged particles; and magnetic forces exerted *by* electric current running through a wire. Because permanent magnets are actually rather complicated, I'll start with the simpler case of a magnetic force exerted on a moving charged particle, by a current-carrying wire.

2.1 A Charge Moving Parallel to a Wire

To begin, I would like to construct a *model* of what is happening at the microscopic level when current flows through a wire. This model may not be realistic in all its details, but it does turn out to contain all the features of real currents in real wires that are essential for our purpose.

Suppose you have a long wire stretched out horizontally in front of you, with a current flowing through it toward the right (see figure 2.1). In principle, a current to the right could be caused by the flow of positive charges to the right, or negative charges to the left, or some combination of both. (The current is conventionally defined as the *net* amount of positive charge that passes a fixed point, moving from left to right in this case. Positive charge flowing to the left counts as negative current, while the flow of negative charge counts like positive charge moving in the opposite direction.) You may know that in real metal wires it is actually negatively charged particles that do the flowing. To avoid having to keep track of lots of minus signs, however, I will take the moving charges to be positive in this simplified model.

So we have a bunch of positive charges, flowing to the right. Let's say that each of them carries charge q and moves with speed v, and that the average separation between adjacent charges is ℓ. Meanwhile, assuming that the wire is electrically neutral, it must contain an equal amount of negative charge. Again for simplicity, let's assume that each negatively charged particle has charge $-q$, and that the negative charges are evenly spaced. Then the average distance between the negative charges must also be equal to ℓ, as shown in figure 2.1.

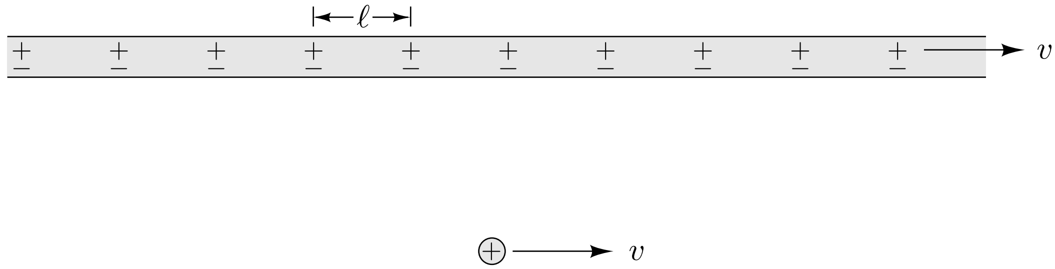

Figure 2.1. A simple model of what goes on in a wire. Positive charges, evenly spaced, move to the right, while an equal number of negative charges remain at rest. If the wire is electrically neutral, the distance between adjacent positive charges must be the same as the distance between adjacent negative charges. Meanwhile, a positive test charge Q moves to the right with the same speed as the positive charges in the wire.

Suppose now that there is also a positively charged particle, with charge Q, outside the wire and traveling (initially) in a direction parallel to the current. I'll refer to this particle as the *test charge*. To keep things as simple as possible, let's take the speed of the test charge to be v, the same as the speed of the moving charges in the wire. As you can readily demonstrate in an experiment, the test charge should experience a magnetic force. How can we derive this result on the basis of what we already know?

So far in these notes I've said nothing about the situation where *both* the source charges and the test charge are moving. The key to handling this case is to consider what happens in a different reference frame, where things are simpler. Consider a reference frame where the test charge is at rest, at least initially (before it starts bending). I will refer to this reference frame as the *test charge frame* and the original reference frame as the *lab frame*. In the test charge frame the only possible force is the electrostatic force $Q\vec{E}$, since the electric field is defined as the force exerted on a unit positive test charge that is at rest.

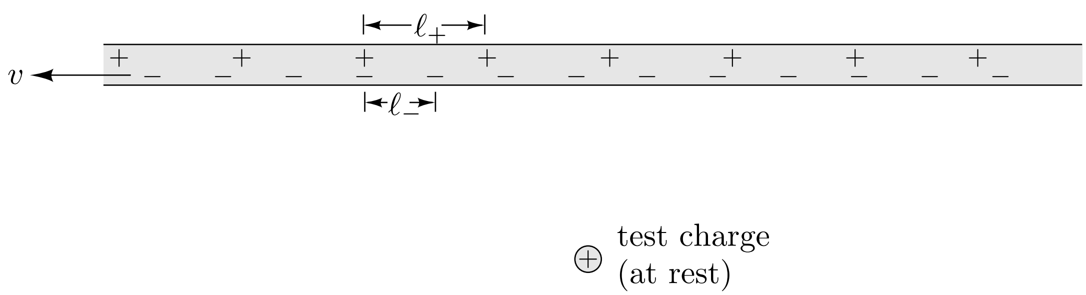

Figure 2.2. The same situation as in figure 2.1, but viewed from the reference frame in which the test charge is initially at rest. Here the positive charges in the wire are at rest while the negative charges in the wire are moving to the left. The distance between the negative charges is length-contracted relative to the lab frame, while the distance between the positive charges is un-length-contracted, so the wire carries a net negative charge.

The situation in the test charge frame is shown in figure 2.2. The positive charges in the wire are also at rest, but the negative charges in the wire are moving to the left with speed v. Now think about length contraction. The negative charges are at rest in the lab

frame but moving in the test charge frame, so the distance between them is *smaller* in the test charge frame than in the lab frame. To be precise, their average separation is now

$$\ell_- = \ell\sqrt{1-(v/c)^2}. \tag{2.1}$$

On the other hand, the positive charges are at rest in the test charge frame but moving in the lab frame, so the opposite applies to them: The distance between them is *larger* in the test charge frame than in the lab frame. Their separation in the test charge frame is

$$\ell_+ = \frac{\ell}{\sqrt{1-(v/c)^2}}. \tag{2.2}$$

Both of these effects give the wire a net *negative* charge in the test charge frame. A negatively charged wire exerts an attractive electrostatic force on a positively charged particle, so our test charge is attracted toward the wire, and begins moving toward it.

Relativistic length contraction thus seems to account for the attractive force between parallel currents, at least in this simplified special case. What if the currents are in opposite directions? Consider next the same situation, but with the test charge moving to the left (see figure 2.3). This situation will be more difficult to analyze, since neither the positive nor the negative charges are at rest in the test charge's reference frame. The negative charges are moving with speed v in the test charge frame, so the distance between them is again

$$\ell_- = \ell\sqrt{1-(v/c)^2}. \tag{2.3}$$

To work out the distance between the positive charges, however, is more difficult. First we must ask what their speed is in the test charge frame. You might think the answer is $2v$, since their speed in the lab frame is v and the test charge frame moves with speed v relative to the lab frame. In fact, the theory of relativity says that the answer is somewhat less than $2v$, although the difference is negligible if v is much less than the speed of light. Let us make this assumption for simplicity. Then the distance between the positive charges is contracted by a factor of $\sqrt{1-(2v/c)^2}$ relative to its value in *their* rest frame, which in turn (as we saw in equation (2.2)) is larger than ℓ by a factor of $\sqrt{1-(v/c)^2}$. Putting this all together, we have

$$\ell_+ = \frac{\ell}{\sqrt{1-(v/c)^2}}\sqrt{1-(2v/c)^2}. \tag{2.4}$$

Now the question is, which is larger—ℓ_- or ℓ_+? The easiest way to answer this question is to again use the assumption that $v \ll c$, and apply the binomial approximation

$$(1+x)^p \approx 1+px \qquad \text{when } |x| \ll 1 \tag{2.5}$$

to both equation (2.1) and equation (2.4). The expression for ℓ_- then becomes

$$\ell_- = \ell\Big(1-\frac{v^2}{c^2}\Big)^{1/2} \approx \ell\Big(1-\frac{v^2}{2c^2}\Big). \tag{2.6}$$

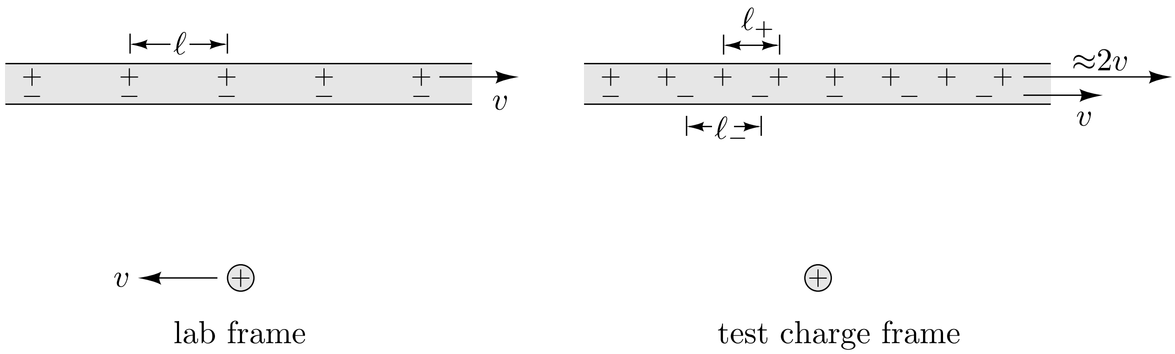

Figure 2.3. The same as figures 2.1 and 2.2, but now the test charge is moving to the left, and the wire is positively charged in the test charge reference frame.

In the expression (2.4) for ℓ_+ we must apply the approximation to both the numerator and the denominator, and multiply out all the terms:

$$\begin{aligned}\ell_+ &= \ell\Big(1-\frac{v^2}{c^2}\Big)^{-1/2}\Big(1-\frac{4v^2}{c^2}\Big)^{1/2}\\ &\approx \ell\Big(1+\frac{v^2}{2c^2}\Big)\Big(1-\frac{4v^2}{2c^2}\Big)\\ &= \ell\Big(1-\frac{3v^2}{2c^2}-\frac{v^4}{c^4}\Big).\end{aligned} \tag{2.7}$$

Comparing expressions (2.6) and (2.7), you can now see that ℓ_+ is less than ℓ_-. Thus the wire is *positively* charged in the test charge frame.

I hope you can see now how length contraction accounts for both the attraction of parallel currents and the repulsion of antiparallel currents. Admittedly, we have made some simplifying assumptions, especially in taking the speed of the charges inside the wire to be the same as the speed of the test charge. However, the same qualitative conclusion holds for any values of these speeds, as well as for the case where the current in the wire is the result of the flow of negative rather than positive charges. The only property of the wire that matters is the net current. The magnitude of the force is determined by the strength of the current in the wire, the speed of the outside particle, and the distance between them.

Exercise 2.1. Consider again the case where the test charge is moving in the same direction as the charges in the wire, but suppose now that the test charge is moving more slowly. Argue qualitatively that the test charge again feels an attractive force, but that this force is weaker in magnitude than before.

Exercise 2.2. Analyze the case where the test charge and the charges in the wire are moving in opposite directions, with different speeds. Show that the direction of the force on the test charge is the same as in the special case considered above, where the speeds were the same. You may assume that both speeds are much less than the speed of light.

Exercise 2.3. Argue that if the charge of the test charge is doubled, the force exerted on it by the wire is also doubled. (Show this for both directions of motion of the test charge.) What if the test charge is negative?

Exercise 2.4. (For those who have studied relativistic velocity transformations.) Correct equation (2.4) to be exact no matter how large v is, and show without making any approximations that ℓ_- is larger than ℓ_+ in this situation. You will have to do some algebra.

2.2 A Charge Moving Perpendicular to a Wire

Experiments show that a current-carrying wire exerts a force on a moving charged particle not only when the particle is moving parallel to the wire, but also when it is moving directly toward or away from the wire. The qualitative explanation of the force in this case also involves length contraction, but is less direct. Consider the situation shown in figure 2.4, where the wire is the same as in the previous section, but the positive test charge is moving directly toward the wire. Figure 2.5 shows the same situation in the reference frame of the test charge, where the wire is moving straight down. The negative charges in the wire are moving straight down with it, and their electric field is symmetrical from side to side. The positive charges in the wire, however, are moving diagonally in the test charge frame. I claim that, at the location of test charge, the net electric field of these positive charges has a nonzero horizontal component, which points to the left.

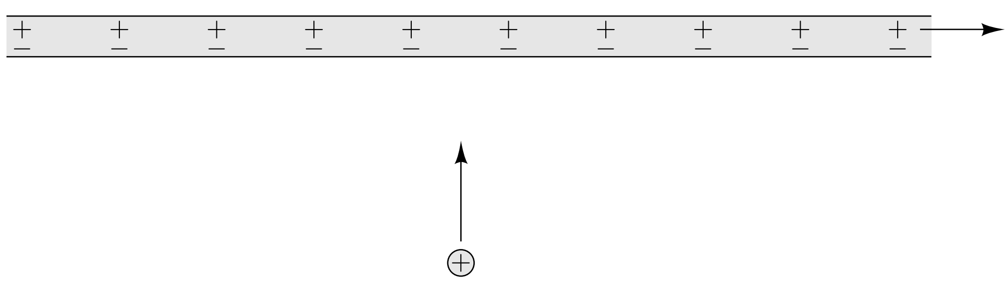

Figure 2.4. Again the wire carries a current of positive charges moving to the right, but now the test charge is moving directly toward it.

Recall from Section 1.2 that the electric field of a moving point charge is not symmetrical, but is instead distorted, being reduced in strength along the direction of motion and intensified in the transverse directions (see figure 1.5). This distortion is ultimately a consequence of length contraction, as discussed in Lesson 1. The distorted fields of two of the individual positive charges are shown in figure 2.5. Notice that at the location of the test charge, the field of the charge on the right is stronger than that of the charge on the left. The same would be true for any other pair of symmetrically placed positive charges in the wire. The net electric field at the test charge's location therefore has a nonzero horizontal component, pointing to the left. (The vertical component of the net field turns

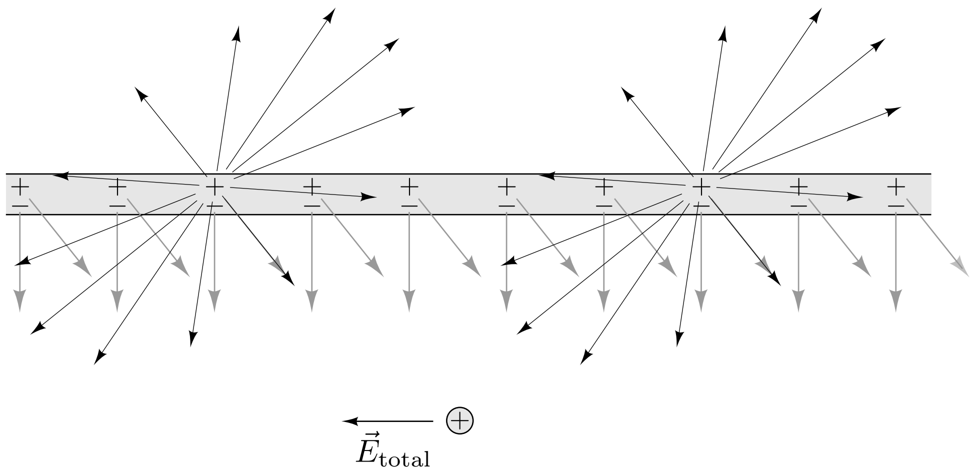

Figure 2.5. The same situation as in figure 2.4, but viewed in the test charge's frame of reference. The negative charges in the wire are moving straight down, while the positive charges are moving diagonally. Since the electric field of a moving charge is weaker along the direction of its motion and stronger in the transverse directions, there is a net horizontal electric field at the location of the test charge.

out to be zero when the negative charges are taken into account, although this is not easy to see.)

Exercise 2.5. Draw an analogous sketch for the case where the test charge is moving directly *away* from the wire, and argue that in this case there is a net force on it toward the right.

2.3 The Lorentz Force Law

Figure 10.9 summarizes the qualitative results of the previous two sections. A test charge located near a current-carrying wire feels a velocity-dependent "magnetic" force. When the current flows to the right and the test charge is below the wire and positively charged, this force is in a direction 90° counter-clockwise from the direction of motion. (Although I haven't proved it, this is true even if the test charge is moving in a diagonal direction.)

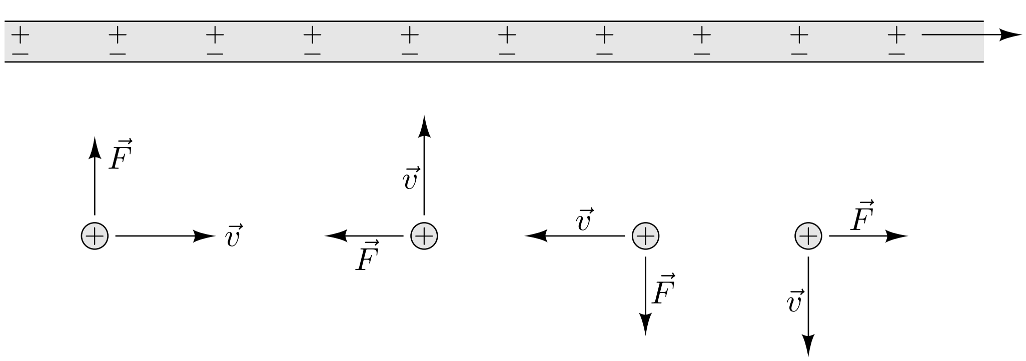

Figure 2.6. For each of the four directions of motion shown, the direction of the force on the test charge is 90° counter-clockwise from the direction of motion.

In this section I want to give you a mathematical formula that summarizes this information. To do so I must introduce a new abstract entity, the *magnetic field*. Like the electric field, the magnetic field is a vector function—an infinite number of little arrows, one at every point in space. The idea is to assume that the wire sets up a magnetic field in the space around it, and that the test charge then responds only to the value of the magnetic field at its own location. Using the idea of a magnetic field, we can avoid transforming back and forth between reference frames every time we want to calculate a magnetic force. The conventional symbol for the magnetic field vector is $\vec{B}$, probably because M already stands for too many other things.

Which way should the magnetic field point in the example of figure 2.6? It would be simple if the magnetic field always pointed in the direction of the force, but there's absolutely no way I can arrange that here, since the force could be in all sorts of different directions depending on the direction the test charge is moving. At this point, therefore, I will resort to mathematical trickery: I will define the magnetic field to point in the one direction that has nothing to do with either the force or the velocity: perpendicular to the page! Actually there are two such directions, up out of the page and down into the page, and I could equally well choose either one. Following tradition, I will choose the field to point *down* into the page in this case. What good does this definition do? I can now summarize our knowledge of magnetic forces by saying that the magnetic force on a test charge always points in a direction perpendicular to both its velocity and the magnetic field. There are two such directions, and for a positive test charge you can find the correct one using the *right-hand rule*: Place the field vector $\vec{B}$ and the velocity vector $\vec{v}$ tail-to-tail, as in figure 2.7. With your right fist near the tails of the vectors, curl your fingers from the $\vec{v}$ vector to the $\vec{B}$ vector, going around whichever way is shorter. Your thumb then points in the direction of the force. Please stop now to try out the right-hand rule on the vectors in figure 2.7. Then go back to figure 2.6, and check that, with $\vec{B}$ pointing down into the page, the right-hand rule gives you the correct direction for the force in each case shown.

Figure 2.7. To apply the right-hand rule, place the velocity vector and the magnetic field vector tail-to-tail, as shown. Since this particle is positively charged, the magnetic force on it points out of the page in (a) and into the page in (b).

I still haven't given you a mathematical formula for the magnetic force on a particle. So let me give it to you now, without further delay:

$$\vec{F}_{\text{magnetic}} = q\vec{v} \times \vec{B}. \tag{2.8}$$

Here q is the charge on the particle, $\vec{v}$ is its velocity vector, and $\vec{B}$ is the magnetic field vector at its location. The symbol $\times$ denotes a *cross product*, similar to that used in

defining angular momentum. The cross product of two vectors is another vector, whose direction is given by the right-hand rule and whose magnitude is given by

$$|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}| \sin\theta, \tag{2.9}$$

where θ is the angle between the two vectors when they are placed tail to tail.

According to equation (2.8), the SI unit of magnetic field should be a newton per coulomb per (meter per second). This unit is called the *tesla*, abbreviated T:

$$1 \text{ tesla} \equiv 1 \, \frac{\text{newton} \cdot \text{second}}{\text{coulomb} \cdot \text{meter}}. \tag{2.10}$$

The field near a powerful laboratory magnet might have a strength of a few tesla. The earth's magnetic field has a strength of less than 10^{-4} tesla. To measure the strength of a magnetic field you must measure the force on a test charge moving with a known velocity.

Equation (2.8) says that the magnetic force on a particle is stronger when it is moving faster, or when it carries a greater charge, or when its velocity is more nearly perpendicular to the direction of the magnetic field. The first two of these facts follow from the arguments of Sections 2.1 and 2.2. The third fact is something I haven't discussed so far, since it involves thinking about what happens when the particle moves parallel to the direction of the field, which would be down into the page in figure 2.6. If you're good at visualizing things in three dimensions, try to imagine what the electric field of the wire then looks like in the test charge frame, and convince yourself that it's plausible for there to be no force in that case.

Exercise 2.6. Suppose that in some location there is a magnetic field that points in the x direction. A particle moving initially in the y direction bends toward the z direction. Does this particle have a positive or negative electric charge?

If both electric and magnetic fields are present at the location of a test charge in some reference frame, then the total electromagnetic force on it is given by

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B}). \tag{2.11}$$

This equation is called the *Lorentz force law.* Its logical status is complex. On one hand, it is the definition of both $\vec{E}$ and $\vec{B}$: To measure the electric and magnetic fields, you must measure the forces on various test charges, at rest or moving with various velocities, and see what values of $\vec{E}$ and $\vec{B}$ work in this equation. On the other hand, it is not at all obvious *a priori* that it is possible to find values of $\vec{E}$ and $\vec{B}$ that work for *all* of the infinitely many possible velocity vectors of the test charge. Thus the Lorentz force law makes a powerful statement about nature: To determine the electromagnetic force on any particle that passes through a point, no matter what its velocity, all you have to know are six numbers, namely the three components of $\vec{E}$ and the three components of $\vec{B}$.

Lesson 3

Calculating the Magnetic Field

3.1 The Magnetic Field of a Wire

In this section I will turn the qualitative argument of Section 2.1 into a quantitative one, and calculate the magnitude of the force exerted by a current-carrying wire on a moving charge. This is equivalent to calculating the magnetic field produced by the wire. As usual, I will derive the relevant formulas in a special, simple situation, then assert that the final results are true more generally.

Consider again the situation shown in figures 2.1 and 2.2. The latter figure, showing the situation in the reference frame of the test charge, is reproduced in figure 3.1. The positive charges in the wire, each with charge q, are at rest in this frame, while the negative charges, each with charge $-q$, are moving to the left with speed v. I have already argued that the average distance between the negative charges in this frame is length-contracted to

$$\ell_- = \ell\sqrt{1-(v/c)^2}, \tag{3.1}$$

where ℓ is the distance between them in the lab frame. Similarly, the distance between the positive charges is un-length-contracted:

$$\ell_+ = \frac{\ell}{\sqrt{1-(v/c)^2}}. \tag{3.2}$$

Both of these effects give the wire a net negative charge in the test charge frame, so that it exerts an attractive force on the test charge.

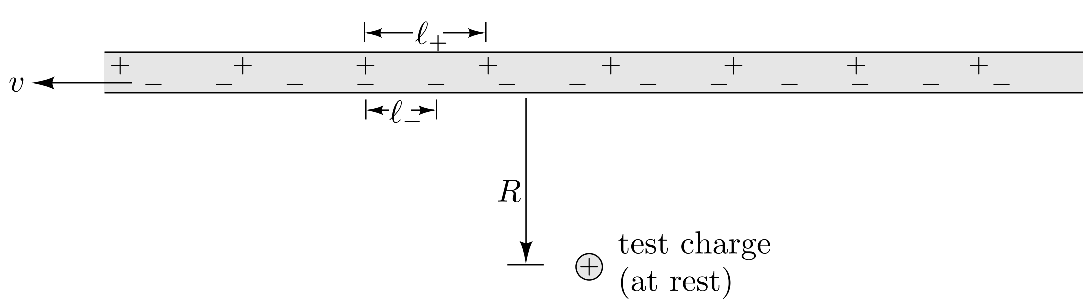

Figure 3.1. A reproduction of figure 2.2, showing our model of a wire with a test charge outside, as viewed from the test charge frame. The particles in the wire have charges $\pm q$, while the test charge has charge Q.

The next step is to be precise about how strong this attractive force is. To do this we must calculate the strength of the electric field at the location of the test charge. And to do *this*, we must know exactly how much charge is on the wire. I haven't told you how long the wire is, and its total length turns out not to matter anyway, as does its total charge. What matters is the charge *per unit length*, or the linear charge density, λ:

$$\lambda \equiv \text{charge per unit length.} \tag{3.3}$$

The total value of the linear charge density is equal to the sum of the linear charge densities of the postive and negative charges separately, taking signs into account. The linear charge density of just the positive charges is

$$\lambda_+ = \frac{q}{\ell_+} = \frac{q\sqrt{1-(v/c)^2}}{\ell}, \tag{3.4}$$

while the linear charge density of just the negative charges is

$$\lambda_- = \frac{-q}{\ell_-} = \frac{-q}{\ell\sqrt{1-(v/c)^2}}. \tag{3.5}$$

The total linear charge density is therefore

$$\lambda = \lambda_+ + \lambda_- = \frac{q}{\ell}\left(\sqrt{1-(v/c)^2} - \frac{1}{\sqrt{1-(v/c)^2}}\right). \tag{3.6}$$

With a bit of algebra you can show that this expression reduces to

$$\lambda = \frac{q}{\ell}\frac{-(v/c)^2}{\sqrt{1-(v/c)^2}}. \tag{3.7}$$

Notice that the net charge is *negative*, indicating that the electric field points *toward* the wire.

I would now like to make two simplifications to equation (3.7). First, I'll eliminate q and ℓ in favor of I, the current flowing through the wire (as measured in the lab frame). Using the definition of the current, you should be able to show that

$$I = \frac{qv}{\ell}. \tag{3.8}$$

Please stop now to derive this equation. Then combine it with equation (3.7), to show that the charge density of the wire is

$$\lambda = \frac{-Iv}{c^2\sqrt{1-(v/c)^2}}. \tag{3.9}$$

Second, I'll assume from here on that everything is moving slowly compared to the speed of light: $v \ll c$. Then the term $(v/c)^2$ in the denominator can be neglected, and we're left with

$$\lambda = \frac{-vI}{c^2}. \tag{3.10}$$

Although we've derived this formula for the special case where the velocity of the positive charges in the wire is the same as the velocity of the test charge, it turns out to be true much more generally: Just take I to represent the net current in the wire (no matter how it arises—from motion of positive charges, or negative, or both, at any speed) and v to represent the velocity of the test charge. If the current or the test charge is going to the left, take I or v to be negative.

Exercise 3.1. Suppose that the current arises from a flow of positive charges to the right, moving with speed u, where $u \neq v$. Derive formula (3.10). Assume that $v \ll c$ and $u \ll c$, so that the velocity of the positive charges in the test charge frame is simply $u - v$. (If you've studied relativistic velocity transformations, you can derive equation (3.9) in general, for arbitrary values of u and v.)

Now that we know how much charge is on the wire in the test charge frame, the next step is to calculate the electric field produced by this charge. You should already know how to use Gauss's law to calculate the electric field of a long line of charges that are *at rest*. The same method and the same result will apply here, but we should be careful that all the same assumptions are still valid.

So consider a Gaussian cylinder, of radius R and arbitrary length, centered on the wire (see figure 3.2). To find the field using Gauss's law we must first argue from symmetry that $\vec{E}$ points directly toward the wire. This is not obvious here, since there is a fundamental asymmetry in the motion of the charge distribution—the negative charges are moving to the *left*. However, the electric field of each of these negative charges looks like figure 1.5 (but with the direction of the arrows reversed), and you can see there that this field *is* symmetrical from front to back. When a bunch of fields like this are all lined up in a row and added together, the horizontal components will cancel out due to this symmetry, leaving only a radial component, pointing directly toward the wire.

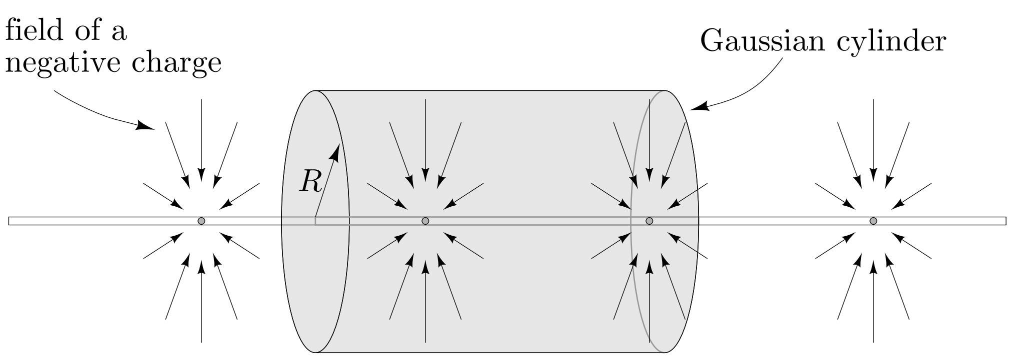

Figure 3.2. To find the electric field in the test charge frame, use a Gaussian cylinder. The individual fields of the moving charges are symmetrical from front to back, so the net field of the wire points directly away from it.

Because the same symmetry arguments apply here, and because Gauss's law is true for moving charges as well as stationary ones, the relation between the linear charge density and the electric field is exactly the same here as in electrostatics:

$$|\vec{E}(R)| = \frac{1}{2\pi\epsilon_0}\frac{|\lambda|}{R}. \tag{3.11}$$

(This formula is exact in the limit where the wire is very long compared to R, and the test charge is far from either end; otherwise it's only approximate.) In our case the charge density is given by equation (3.10), so the magnitude of the electric field is

$$|\vec{E}(R)| = \frac{vI}{2\pi\epsilon_0 c^2 R}. \tag{3.12}$$

Since the wire is negatively charged, the field points toward it. To find the force on the test charge Q, multiply the field by the charge:

$$|\vec{F}| = Q|\vec{E}| = \frac{QvI}{2\pi\epsilon_0 c^2 R}. \tag{3.13}$$

This is the force in the reference frame of the test charge. But if the speed of the test charge is small compared to the speed of light, then the force is the same in both reference frames, so this expression gives the force in the lab frame as well.

In the lab frame, however, there is no electric field (since the wire is electrically neutral), so this force must be due to a magnetic field. In the previous lesson I defined the magnetic field to point into the page in this case, so that the magnetic force on the test charge is given by the general formula

$$\vec{F}_{\text{magnetic}} = Q\vec{v} \times \vec{B}. \tag{3.14}$$

Since $\vec{v}$ and $\vec{B}$ are perpendicular in our situation, the magnitude of the magnetic force must equal simply $Qv|\vec{B}|$. Comparing this expression with equation (3.13), we find that the magnitude of the magnetic field of the wire must be

$$|\vec{B}| = \frac{I}{2\pi\epsilon_0 c^2 R}. \tag{3.15}$$

This is the final result of this section. The magnetic field strength of a long straight wire falls off with distance in proportion to $1/R$, as a result of the similar behavior of the electric field of a line of charge (equation (3.11)). A sketch of the magnetic field of a wire is shown in figure 3.3. To remember the direction of the field of a wire you can use another *right-hand rule*: Point the thumb of your right hand along the direction of the current; then your fingers curl around in the direction of the field.

What if a current-carrying wire is not "long" and "straight"? Then the wire still produces a magnetic field, but the formula for the field is much more complicated. Qualitatively, it is always true that the field due to a small piece of the wire looks more or less like figure 3.3, so it's not hard in general to at least figure out the *direction* of the field of a curved wire. Quantitative formulas for such fields tend to be quite complicated, though, and are beyond the scope of these notes.

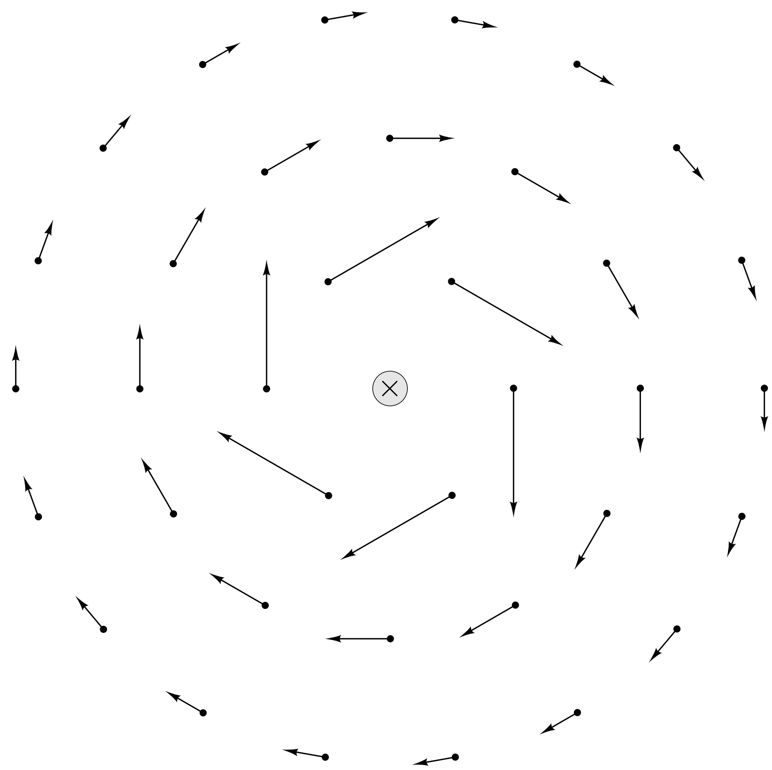

Figure 3.3. A rough sketch of the magnetic field of a long straight wire, shown in cross-section. The wire is in the middle of the figure, and the current is flowing away from you, into the page. The strength of the field falls off as $1/R$, while the direction is given by the right-hand rule.

3.2 The Force Between Two Wires

We are now almost ready to perform a quantitative experimental test of all these theoretical derivations. In particular, it would be nice to test the formula (3.15) for the magnetic field of a current-carrying wire. To do so, we need a convenient means of measuring the strength of a magnetic field. In principle, this could be done by sending a test charge through at some known velocity, and measuring how much it bends. In practice, however, this method is feasible only for microscopic test charges, such as electrons, and it is quite difficult to measure the velocity of an electron directly (without using a magnetic field whose strength is known). A better method is to introduce a second current-carrying wire, parallel to the first. The field due to one wire then exerts a force on the other wire, which can be measured with little difficulty.

We already have a formula for the magnetic field produced by one of the two wires in such an experiment. Now we need a formula for the force exerted on the other wire by this magnetic field. (A wire feels no force due to its own magnetic field.) So consider the situation shown in figure 3.4. There is a magnetic field $\vec{B}$ pointing into the page, and the

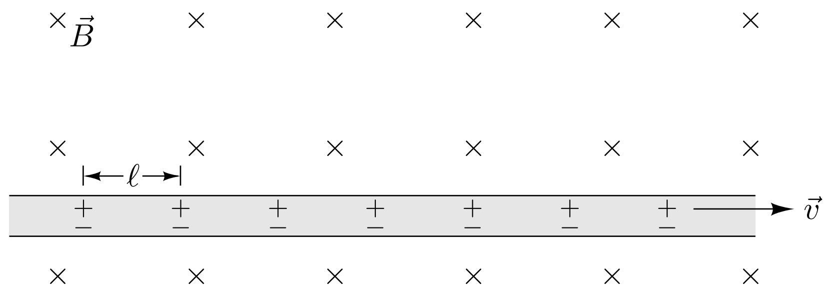

Figure 3.4. A length of wire, with current flowing to the right, is immersed in a magnetic field pointing into the page. The force on the wire points upward, according to the right-hand rule.

wire carries a current I that flows toward the right. To be specific, I'm assuming that the current is carried by positive charges q, traveling at speed $\vec{v}$ and separated by a distance ℓ.

The force on each of the positive charges in the wire points upward, and has magnitude

$$|\vec{F}_q| = q|\vec{v} \times \vec{B}| = q|\vec{v}||\vec{B}|, \tag{3.16}$$

where in the second equality I'm using the fact that $\vec{v}$ and $\vec{B}$ are perpendicular. The *total* force on the wire depends on how many of these charges there are, or equivalently, on how long the wire is. Let's compute the force on a segment of the wire that has length L. Then the number of moving charges in it equals L/ℓ, so the total force has magnitude

$$|\vec{F}_{\text{total}}| = \frac{q|\vec{v}||\vec{B}|L}{\ell}. \tag{3.17}$$

Now the combination $q|\vec{v}|/\ell$ is equal to the current, just as in equation (3.8). I can therefore rewrite this result as

$$|\vec{F}_{\text{total}}| = IL|\vec{B}|. \tag{3.18}$$

This equation is true for any straight segment of wire in a magnetic field where the current is perpendicular to the field.

> **Question 3.2.** Suppose that the wire were inclined at some other angle relative to the direction of the magnetic field. What would the magnitude of the force be in this case? In what direction would the force act?

Combining equations (3.15) and (3.18), you can now derive a formula for the magnitude of the force between two parallel wires. If the wires both have length L, are separated by a distance R, and carry currents I_1 and I_2, the magnetic force acting on each of them has magnitude

$$|\vec{F}| = \frac{I_1 I_2 L}{2\pi\epsilon_0 c^2 R}. \tag{3.19}$$

Since the formula (3.15) is valid only at points close to the wire and far from either end, equation (3.19) is subject to the restriction that the wires be very close together compared to their lengths: $R \ll L$. The direction of the force is determined by applying *both* right-hand rules: To find the field of one wire, point your thumb in the direction of the current

and look which way your fingers curl; then to find the force on the other wire, curl your fingers from the direction of the current to the direction of the field, and look which way your thumb points. You should find that parallel currents attract and anti-parallel currents repel.

It is not hard to test equation (3.19) experimentally. One wire is normally held stationary, while the other is attached to a balance of some sort so the force on it can be "weighed". If you do the experiment carefully, you should find that the prediction is precisely correct. Stop a moment and think about what this means. These last three chapters have been a long sequence of logical steps leading from our earlier knowledge of electrostatics to a prediction of the strength of the magnetic force between two wires. The key ingredient throughout these chapters has been relativistic length contraction—one of the most counter-intuitive ideas in all of physics. And yet here we see its prediction verified, quantitatively, in a phenomenon that is easy to produce using commonplace electrical equipment.

Exercise 3.3. An experiment to measure the magnetic force between two wires might use wires that are 26 cm long, separated by 1/2 cm, with 10 amperes flowing through each. Calculate the magnitude of the magnetic force in newtons, and also the mass, in grams, whose weight is equal to this force.

Although the magnetic force between two wires is usually fairly small, it's a wonder that we can measure it at all. It turns out that the electrons that flow through a wire move *very* slowly, so the relativistic contraction in the distance between them must be tiny indeed. How can such a tiny effect show up in such a simple measurement? The compensating factor is the enormous number of electrons in the wire, together with the intrinsic strength of the electrostatic force.

Exercise 3.4. Consider a length of copper wire one meter long and one millimeter in diameter. By looking up the necessary data, calculate the number of copper atoms in the wire. (The density of copper is 9.0 g/cm^3.) Suppose each atom contributes one electron to the current flowing through the wire. How many coulombs of charge do these electrons amount to? Is this a lot of charge? Suppose that the magnitude of the current is 1 ampere. What is the average speed of the moving electrons? By what fraction is the distance between them length-contracted?

3.3 A Historical Digression

Historically, magnetic forces between wires were discovered and measured long before anyone even dreamed of length contraction. These measurements were made very shortly after Oersted's 1820 discovery that electric currents cause magnetic forces. Within a few years, physicists such as Biot, Savart, and Ampere had formulated laws for calculating the magnetic field around any arrangement of wires carrying steady currents. They did not, of course, write equations like (3.15) and (3.19) in terms of the speed of light, since

there was no obvious relation between light and magnetism. Even in modern notation, equation (3.15), for instance, is normally written in the form

$$|\vec{B}| = \frac{\mu_0 I}{2\pi R}, \tag{3.20}$$

where μ_0 is a constant, equal to $1/\epsilon_0 c^2$. By measuring the strength of the force between two wires, physicists effectively measured the value of μ_0, treating it as an independent fundamental constant of physics. (Alternatively, in some unit systems, μ_0 or its equivalent was taken as a defined quantity, leaving ϵ_0 as the quantity to be measured in the laboratory.)

In the decades following 1820, physicists (most notably Faraday) investigated the more complex phenomena associated with time-dependent currents (such as the alternating, back-and-forth current now used in household wires). They discovered that in these situations, the electric and magnetic fields are even more intimately linked to each other. Eventually, around 1865, James Clerk Maxwell formulated a complete set of mathematical equations that allowed one to calculate the electric and magnetic fields generated by *any* distribution of charges and currents, undergoing *any* sort of motion. One of Maxwell's equations is none other than Gauss's law; another says that the flux of the *magnetic* field through any closed surface must equal zero. The remaining equations involve a concept analogous to flux, called *circulation*, which expresses the rotational character of a field like that shown in figure 3.3. Maxwell's equations are treated in more detail in most introductory physics texts.

One of the immediate consequences of Maxwell's equations is that under some circumstances, a piece of the electric/magnetic field can essentially break free of the charge that produced it and travel on its own, as a wave, to some distant place. Maxwell calculated the speed at which these "electromagnetic waves" ought to travel, and found it to be, in modern notation, $1/\sqrt{\epsilon_0 \mu_0}$. Noticing that this quantity was numerically close to 3×10^8 meters per second, he concluded that light itself is an electromagnetic wave. Here is a quote from his 1865 paper:

> By the electromagnetic experiments of MM. Weber and Kohlrausch,
>
> $$v = 310\,740\,000 \text{ metres per second}$$
>
> is the number of electrostatic units in one electromagnetic unit of electricity [i.e., $1/\sqrt{\epsilon_0 \mu_0}$], and this, according to our result, should be equal to the velocity of light in air or vacuum.
>
> The velocity of light in air, by M. Fizeau's experiments, is
>
> $$V = 314\,858\,000;$$
>
> according to the more accurate experiments of M. Foucault,
>
> $$V = 298\,000\,000.$$

The velocity of light in the space surrounding the earth, deduced from the coefficient of aberration and the received value of the radius of the earth's orbit, is

$$V = 308\,000\,000.$$

Hence the velocity of light deduced from experiment agrees sufficiently well with the value of v deduced from the only set of experiments we as yet possess. The value of v was determined by measuring the electromotive force with which a condenser of known capacity was charged, and then discharging the condenser through a galvanometer, so as to measure the quantity of electricity in it in electromagnetic measure. The only use of light in the experiment was to see the instruments. The value of V found by M. Foucault was obtained by determining the angle through which a revolving mirror turned, while the light reflected from it went and returned along a measured course. No use whatever was made of electricity or magnetism.

The agreement of the results seems to show that light and magnetism are affections of the same substance, and that light is an electromagnetic disturbance propagated through the field according to electromagnetic laws.

Over the next 40 years, experiments continued to support Maxwell's theory. But no one fully understood it until Einstein, who noticed that although an electrical phenomenon in one reference frame might appear to be a magnetic phenomenon in another frame, the same physical phenomenon occurs in both frames. This observation led him to conjecture that the laws of physics, even those of electromagnetism, might be the same in all inertial reference frames, and this conjecture led him to his special theory of relativity in 1905.

3.4 The Forces on a Relativistic Charge

[This is an optional section for students who have studied relativistic dynamics. The goal is to derive exact formulas for the electric and magnetic forces on a charged particle moving at a speed comparable to the speed of light. For a fuller discussion see Purcell, Sections 5.8 and 5.9.]

Exercise 3.5. We already know how a nonrelativistic charged particle responds to an electric field: its (Newtonian) momentum changes according to the equation

$$\frac{d\vec{p}}{dt} = Q\vec{E}, \tag{3.21}$$

where Q is its charge and $\vec{E}$ is the field. How shall we generalize this equation to the case where the charge is moving at relativistic speed? To answer this question, consider a charge Q moving (initially) in the x direction with speed v. It then passes through a region (such as that between a pair of capacitor plates) of length L_0 in which there is a constant electric field $\vec{E}$ pointing in the y direction (see figure 3.5). The time that it spends in this region is $t = L_0/v$. Analyze this situation in the

reference frame of the charged particle: find the field strength in this frame, and use equation (3.21) (since the particle is now nonrelativistic) to compute the amount Δy by which the particle has been deflected from its original trajectory by the time it leaves the region. Argue in one sentence that this deflection must be the same in both reference frames. Then working again in the lab frame, show that you will obtain the proper deflection if you use equation (3.21), but with $\vec{p}$ replaced by the relativistic momentum,

$$\vec{p} = \frac{m\vec{v}}{\sqrt{1-(v/c)^2}} \qquad \text{(relativistic).} \tag{3.22}$$

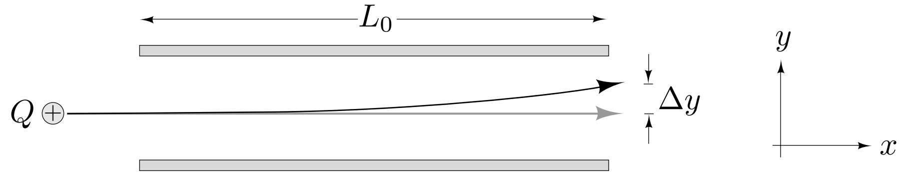

Figure 3.5. A charge Q moving at relativistic speed enters a region with a transverse field, and is deflected (exercise 3.5). By calculating the deflection in the rest frame of the charge, you can derive the correct equation of motion.

Exercise 3.6. Using a method similar to that of the previous exercise, you can derive the correct equation of motion for a relativistic charge subject to a *magnetic* force. Consider a positive test charge moving parallel to a current-carrying wire. As in figure 2.1, assume for simplicity that the current is due to the motion of positive charges in the wire, which have the same velocity as the test charge. Suppose that a segment of the wire with length L_0 is clearly marked for reference. Working in the rest frame of the test charge, calculate the deflection of the test charge as it passes this segment. Then show that, in order to obtain the same deflection in the lab frame, one must use the equation of motion

$$\frac{d\vec{p}}{dt} = Q\vec{v} \times \vec{B}, \tag{3.23}$$

where $\vec{p}$ is the relativistic momentum and the $|\vec{B}|$ is the same as in equation (3.15).

The rotating armatures of every generator and every motor in this age of electricity are steadily proclaiming the truth of the relativity theory to all who have ears to hear.

—Leigh Page

Lesson 4

Radiation

So far these notes have treated phenomena associated with electric charges that are either moving at constant velocity or flowing steadily through wires. The next level of complication involves sudden *changes* in the motion of charge. This is a very rich area for study, which could lead us into alternating currents, time-dependent magnetic fields, and the rest of Maxwell's equations. In the interest of brevity, however, I will go straight to the situation that I find most interesting of all: a point charge that undergoes a sudden change in motion (acceleration), and the electric field produced by this accelerated charge.

4.1 The Field of an Accelerated Charge

Recall from Section 1.2 that when a point charge moves at *constant* velocity, its electric field always points directly away from it, as shown in figure 1.5. (I'll assume for convenience that the point charge is positive.) In light of special relativity theory, this may seem strange, since no information can travel faster than the speed of light. Why then does the field at some faraway place point directly away from where the charge is *now*, rather than from where it was some time ago? Does this imply that information about the motion of the charge travels instantaneously throughout the whole universe? Well, not necessarily. You see, the particle has been traveling at constant velocity, along a predictable course, for some time. So if you're at a faraway place, you could have arranged for the particle to send you information about its position and velocity some time ago, so that when you receive this information you can extrapolate its motion from the past into the present and figure out where it must be by now.

Your scheme for predicting the position of the particle would be ruined, however, if the particle undergoes some acceleration between when it sends you the information and the present. You would think that the particle had continued to travel at constant velocity, and the field at your location would point away from where the particle *would* be now if it had done so, but in fact the particle is not there. For instance, suppose the particle is initially traveling to the right at 1/4 the speed of light, then suddenly bounces off a wall and recoils back to the left at the same speed (see figure 4.1). After one second, the news of the bounce can't have traveled farther than one light-second (300,000 km). If you're closer than one light-second to the location of the bounce then you've already received the news, and the field at your location points away from where the particle is now. But if you're farther than one light-second from the location of the bounce then the news hasn't

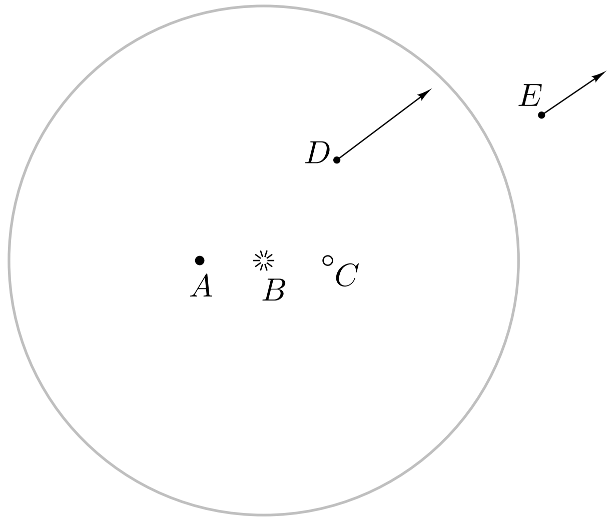

Figure 4.1. A positively charged particle, initially traveling to the right at 1/4 the speed of light, bounces off a wall at point B. The particle is now at point A, but if there had been no bounce it would now be at C. The circle (actually a cross section of a sphere) encloses the region of space where news of the bounce has already arrived; inside this circle (as at D) the electric field points directly away from A. Outside the circle (as at E) the news has not yet arrived, so the field points directly away from C. As time passes the circle expands outward at the speed of light, and points A and C move away from B at 1/4 the speed of light.

reached you yet, and the field at your location points away from where the particle would be now if there had been no bounce.

In this section I will assume that some mechanism like this, or at least equivalent to this, actually operates. We know from special relativity that no information can travel faster than the speed of light. I'll assume the best possible case: that the information travels at precisely the speed of light, but no faster. This assumption, together with Gauss's law, is enough to determine the electric field everywhere around the accelerated charge, and that is the goal of this section.

The complete map of the electric field of an accelerated charge turns out to be fairly complicated. Rather than representing the field as a bunch of arrows (like the two shown in figure 4.1), it is much more convenient to use a more abstract representation in terms of *field lines*. Field lines are continuous lines through space that run parallel to the direction of the electric field. A drawing of the field lines in a region therefore tells us immediately the direction of the electric field, although determining its magnitude is not so easy. A map of the field lines for the situation of figure 4.1 is shown in figure 4.2.

I have not drawn any field lines through the gray spherical shell in figure 4.2, since this is the region that is just in the midst of receiving the news of the particle's acceleration. To determine the direction of the field here, imagine a curved Gaussian "pillbox", indicated by the dashed line in the figure, which straddles the gray shell. (This surface is meant to be symmetrical about the line along which the particle is moving; viewed from along this line, it would be circular.) The Gaussian surface encloses no electric charge, so Gauss's law tells

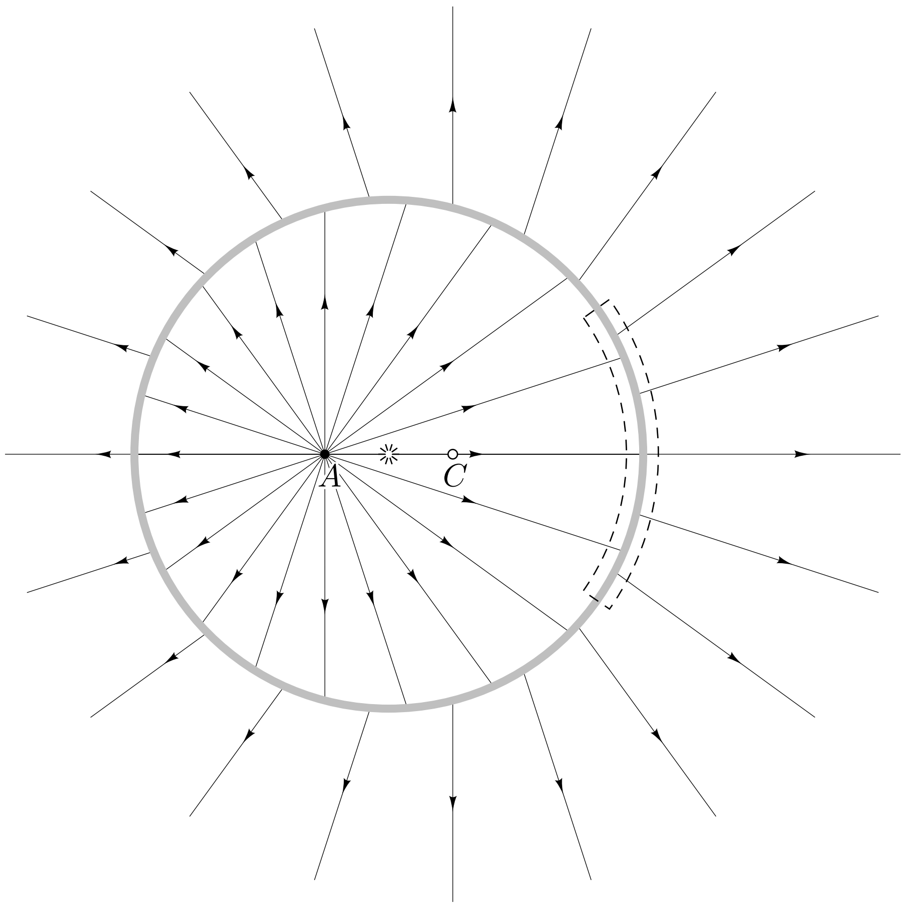

Figure 4.2. A map of the electric field lines for the same situation as in figure 4.1. The direction of the field within the gray spherical shell can be found be considering the flux through the curved Gaussian "pillbox" indicated by the dashed line.

us that the total flux of $\vec{E}$ through it must be zero. Now consider the flux through various parts of the surface. On the outside (right-hand) portion there is a positive flux, while on the inside (left-hand) portion there is a negative flux. But these two contributions to the flux do not cancel each other, since the field is significantly stronger on the outside than on the inside. This is because the field on the outside is that of a point charge located at C, while the field on the inside is that of a point charge located at A, and C is significantly closer than A. The net flux through the inside and outside portions of the surface is therefore positive. To cancel this positive flux, the remaining edges of the pillbox must contribute a negative flux. Thus the electric field within the gray shell must have a nonzero component along the shell, in toward the center of the Gaussian surface. I will refer to this component as the *transverse* field, since it points transverse (i.e., perpendicular) to the purely radial direction of the field on either side.

Exercise 4.1. Use a similar argument to determine the direction of the electric field within the portion of the gray shell on the left side of figure 4.2.

To be more precise about the direction of the field within the gray shell, consider the modified Gaussian surface shown in figure 4.3. Here I have shrunk the outer surface ef

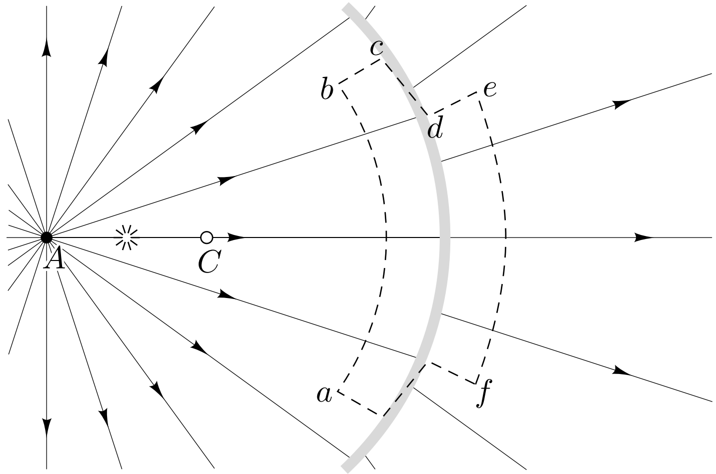

Figure 4.3. Another Gaussian surface applied to the same electric field as in figure 4.2. Since the flux along segment *cd* must be zero, the electric field within the gray shell must be parallel to this segment.

until it subtends the same angle, as viewed from C, that the inner surface ab subtends as viewed from A. Now the fluxes through ab and ef do indeed cancel. Segments bc and de are chosen to be precisely parallel to the field lines in their locations, so there is no flux through these portions of the surface. In order for the total flux to be zero, therefore, the flux must be zero through segment cd as well. This implies that the electric field within the gray shell must be parallel to cd. If you start at A and follow any field line outward, you will turn a sharp corner at the gray shell's inner edge, then make your way along the shell and slowly outward, turning another sharp corner at the outer edge. (The thickness of the gray shell is determined by the duration of the acceleration of the charge.) A complete drawing of the field lines for this particular situation is shown in figure 4.4.

> **Exercise 4.2.** Sketch the field lines for a point charge that undergoes each of the following types of motion. (a) The charge moves to the right at 1/4 the speed of light, then suddenly stops. (b) The charge is initially at rest, then suddenly begins moving to the right at 1/4 the speed of light. (c) The charge is initially moving to the right at 1/2 the speed of light, then suddenly slows down to 1/4 the speed of light without changing direction. (d) The charge is bouncing back and forth, at 1/4 the speed of light, between two walls. (e) The charge is initially moving to the right at 1/4 the speed of light, then makes a sharp 90 degree turn without changing speed.

The transverse portion of the electric field of an accelerated charge is also called the *radiation field*, because as time passes it "radiates" outward in a sphere expanding at the speed of light. If the acceleration of the charged particle is sufficiently great, the radiation field can be quite strong, affecting faraway charges much more than the ordinary radial field of a charge moving at constant velocity. The radiation field can also store a relatively large amount of energy, which is carried away from the charge that created it. In the next section I will justify these claims by deriving a formula for the strength of the radiation field.

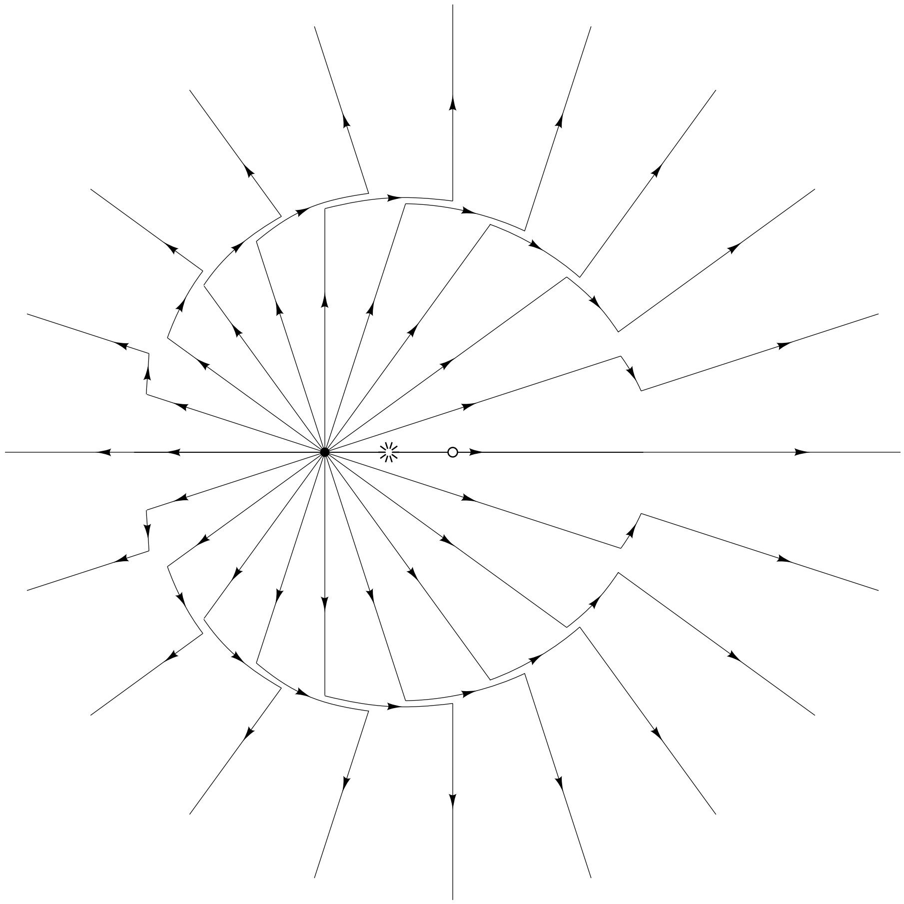

Figure 4.4. A complete sketch of the electric field lines for the situation shown in the preceding figures, including the transverse radiation field created by the acceleration of the charge.

4.2 The Strength of the Radiation Field

To turn the qualitative ideas of the previous section into quantitative formulas, let us consider a somewhat simpler situation, in which a positively charged particle, initially moving to the right, suddenly stops and then remains at rest. Let v_0 be the initial speed of the particle, and let the deceleration begin at time $t = 0$ and end at time $t = t_0$. I'll assume that the acceleration is constant during this time interval; the magnitude of the acceleration is then

$$a = |\vec{a}| = \frac{v_0}{t_0}. \tag{4.1}$$

I'll also assume that v_0 is much less than the speed of light, so that the relativistic compression and stretching of the electric field discussed in Section 1.2 is negligible.

Figure 4.5 shows the situation at some time T, much later than t_0. The "pulse" of radiation is contained in a spherical shell of thickness ct_0 and radius cT. Outside of this shell, the electric field points away from where the particle would have been if it had kept going; that point is a distance v_0T to the right of its actual location. (The distance that

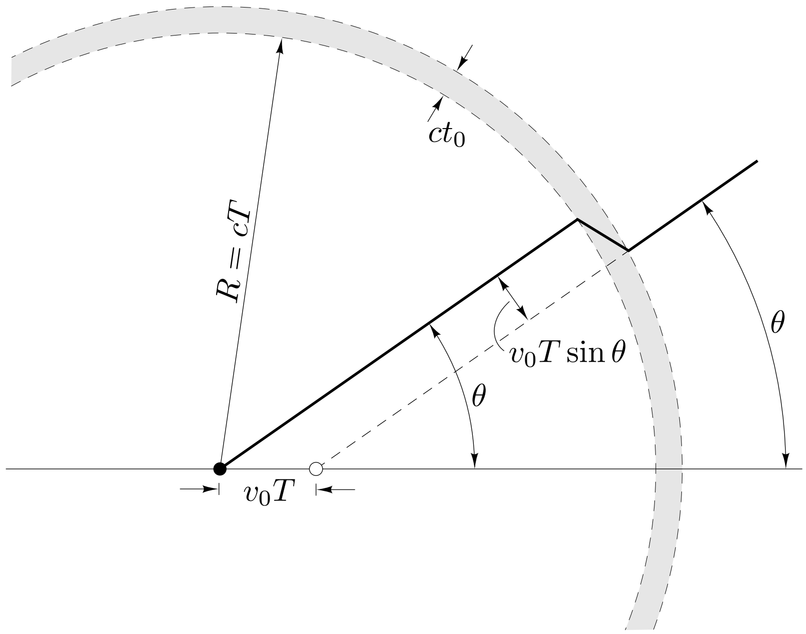

Figure 4.5. Figure for determining the strength of the electric field within the pulse of radiation. For clarity, only a single field line is shown here.

it traveled during the deceleration is negligible on this scale.) A single field line is shown in the figure, coming out at an angle θ from the direction of the particle's motion. There is a sharp kink in this line where it passes through the shell, as discussed in the previous section. We would like to know how strong the electric field is within the shell.

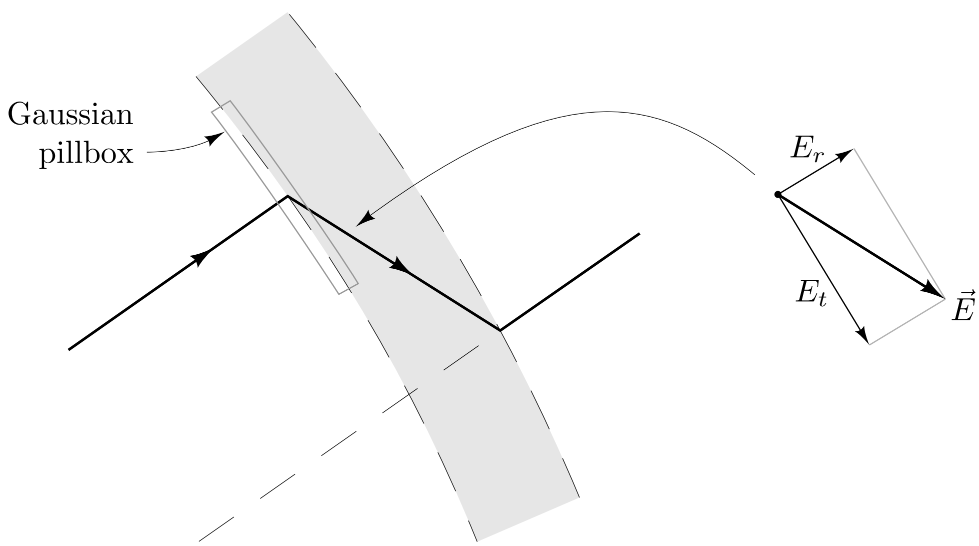

Figure 4.6. Close-up of the kink in the field in figure 4.5. The radial component E_r of the kinked field can be found by applying Gauss's law to the pillbox shown.

Let's break the kinked field up into two components: a radial component E_r that points away from the location of the particle, and a transverse component E_t that points in the perpendicular direction (see figure 4.6). The ratio of these components is determined by the direction of the kink; from figure 4.5 you can see that

$$\frac{E_t}{E_r} = \frac{v_0 T \sin\theta}{c t_0} = \frac{a T \sin\theta}{c}. \tag{4.2}$$

We can find the radial component E_r by applying Gauss's law to a tiny pillbox that straddles the inner surface of the shell (see figure 4.6). Let the sides of the pillbox be infinitesimally short so that the flux through them is negligible. Then since the net flux through the pillbox is zero, the radial component of $\vec{E}$ (that is, the component perpendicular to the top and bottom of the pillbox) must be the same on each side of the shell's inner surface. But inside the sphere of radiation the electric field is given by Coulomb's law. Thus the radial component of the kinked field is

$$E_r = \frac{1}{4\pi\epsilon_0}\frac{q}{R^2}, \tag{4.3}$$

where q is the charge of the particle. Combining equations (4.2) and (4.3) and using the fact that $R = cT$, you should now be able to show that

$$E_t = \frac{qa\sin\theta}{4\pi\epsilon_0 c^2 R}. \tag{4.4}$$

Exercise 4.3. Although I've derived formula (4.4) for the special case where the particle's final velocity is zero, it is true much more generally. To convince yourself of this, consider the case where the particle is *initially* at rest, then receives a sudden kick to the right. Draw a picture analogous to figure 4.5, and follow the same reasoning to arrive at equation (4.4). (Depending on your choice of coordinates, you may find an additional minus sign.)

Equation (4.4) tells us all we need to know about the strength of the pulse of radiation. First, note that the transverse field is proportional to $1/R$, not $1/R^2$. This means that as time goes on and R increases, the transverse field becomes much stronger than the radial field; at very large distances the radial field can be completely neglected and the field is purely transverse. Second, consider the dependence of E_t on the angle θ: It is weakest along the direction of motion ($\theta = 0$ or $180°$) and strongest at right angles to the motion ($\theta = 90°$). Looking back at figure 4.4, we see that the size of the kink in the field is a qualitative indication of the field strength. Finally, notice that the strength of the transverse field is proportional to a, the magnitude of the particle's acceleration. The greater the acceleration, the stronger the pulse of radiation.

This pulse of radiation carries energy. Recall from electrostatics that the energy per unit volume stored in any electric field is proportional to the square of the field strength. In our case, this implies

$$\text{Energy per unit volume} \propto \frac{a^2}{R^2}. \tag{4.5}$$

Since the volume of the spherical shell (the shell itself, not the region it encloses) is proportional to R^2, the total energy it contains does not change as time passes and R increases. Thus when a charged particle accelerates, it loses energy to its surroundings, in an amount proportional to the square of its acceleration. This process is the basic mechanism behind all electromagnetic radiation: visible light and its invisible cousins, from radio waves to gamma rays. Lesson 5 discusses a few applications of this all-important result.

4.3 The Larmor Formula

In this section I will derive a precise formula for the energy radiated by an accelerated charged particle. You've already read the most important part of the derivation, which ended with equation (4.4). The rest is mostly math.

The energy per unit volume stored in any electric field is

$$\text{Energy per unit volume} = \frac{\epsilon_0}{2}|\vec{E}|^2. \tag{4.6}$$

Once the pulse becomes large enough we can neglect the radial component of the field and simply plug in E_t for $|\vec{E}|$. The result is

$$\text{Energy per unit volume} = \frac{q^2 a^2 \sin^2\theta}{32\pi^2\epsilon_0 c^4 R^2}. \tag{4.7}$$

Notice again that this formula is largest when $\theta = 90°$.

If we don't care about the direction in which the energy goes, it is convenient to average equation (4.7) over all directions. I'll do this using a mathematical trick. Introduce a coordinate system with the origin at the center of the sphere and the x axis along the particle's original direction of motion. Then for any point (x, y, z) on the spherical shell, $\cos\theta = x/R$. Using angle brackets $\langle\,\rangle$ to denote an average over all points on the shell, I claim that

$$\langle\sin^2\theta\rangle = \langle 1 - \cos^2\theta\rangle = 1 - \frac{\langle x^2\rangle}{R^2}. \tag{4.8}$$

Now since the origin is at the center of the sphere, you must certainly agree that the average value of x^2 is the same as the average value of y^2 or z^2:

$$\langle x^2\rangle = \langle y^2\rangle = \langle z^2\rangle. \tag{4.9}$$

But this implies that

$$\langle x^2\rangle = \frac{1}{3}\langle x^2 + y^2 + z^2\rangle = \frac{1}{3}\langle R^2\rangle = \frac{R^2}{3}, \tag{4.10}$$

since $x^2 + y^2 + z^2 = R^2$ and R is constant over the whole shell. Combining equations (4.8) and (4.10) gives

$$\langle\sin^2\theta\rangle = 1 - \frac{R^2}{3R^2} = \frac{2}{3}. \tag{4.11}$$

So the average energy per unit volume stored in the transverse electric field is

$$\text{Average energy per unit volume} = \frac{q^2 a^2}{48\pi^2 \epsilon_0 c^4 R^2}. \tag{4.12}$$

To obtain the total energy stored in the transverse electric field, we must multiply equation (4.12) by the volume of the spherical shell. The surface area of the shell is $4\pi R^2$ and its thickness is ct_0, so its volume is the product of these factors. Therefore the total energy is

$$\text{Total energy in electric field} = \frac{q^2 a^2 t_0}{12\pi \epsilon_0 c^3}. \tag{4.13}$$

Notice that the total energy is independent of R; that is, the shell carries away a fixed amount of energy that is not diminished as it expands.

Now, in order to be completely precise, I have to cheat. So far I've discussed only the *electric* field of the accelerated charge. But it turns out that there is also a *magnetic* field, which carries away an equal amount of energy. Since I've omitted so many details about magnetic fields from these notes, I have no way of justifying this claim. An error of a factor of 2 would hardly matter for the applications we'll be considering anyway, but I think it's better to go ahead and put it in for the record. Thus the *total* energy carried away by the pulse of radiation is twice that of equation (4.13), or

$$\text{Total energy in pulse} = \frac{q^2 a^2 t_0}{6\pi \epsilon_0 c^3}. \tag{4.14}$$

It is usually more convenient to divide both sides of this equation by t_0, the duration of the particle's acceleration. The left-hand side then becomes the energy radiated by the particle per unit time, or the *power* given off during the acceleration:

$$\text{Power radiated} = \frac{q^2 a^2}{6\pi \epsilon_0 c^3}. \tag{4.15}$$

This result is called the *Larmor formula*, since it was first derived (using a more difficult method) by J. J. Larmor in 1897. The derivation given here was first published by J. J. Thomson (discoverer of the electron) in 1907. Although I have derived it for the special case where the final velocity of the particle is zero, the Larmor formula is true for any sort of accelerated motion provided that the speed of the particle is always much less than the speed of light.

Lesson 5

Applications of Radiation

5.1 Electromagnetic Waves

In the previous lesson I argued that when a charged particle accelerates, part of its electric field breaks free and travels away at the speed of light, forming a pulse of *electromagnetic radiation*. Often, in practice, charged particles oscillate back and forth continuously, sending off one pulse after another in a periodic pattern. An example of the electric field around an oscillating charge is shown in figure 5.1.

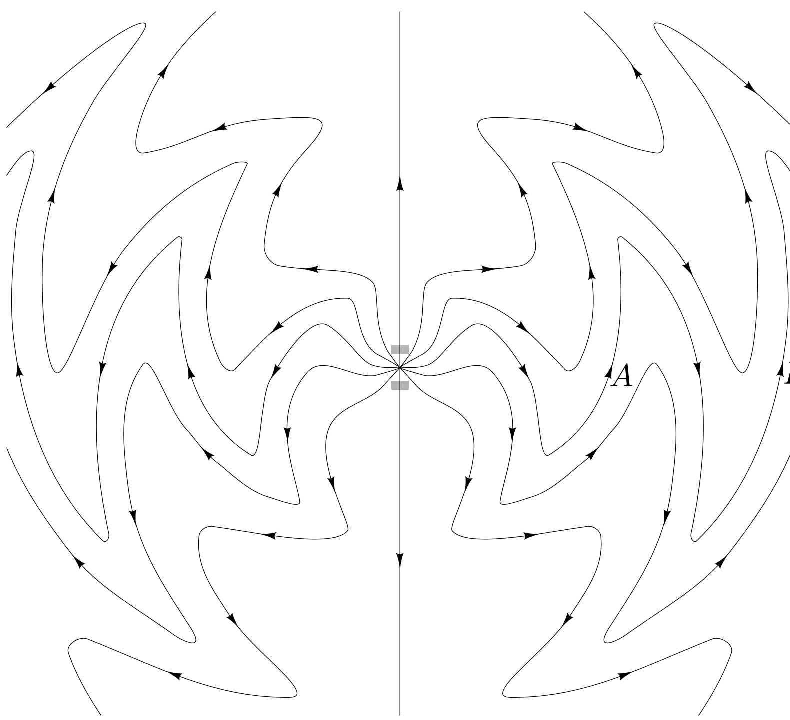

Figure 5.1. A map of the electric field lines around a positively charged particle oscillating sinusoidally, up and down, between the two gray regions near the center. Points A and B are one wavelength apart.

If you follow a straight line out from the charge at the center of figure 5.1, you will find that the field oscillates back and forth in direction. The distance over which the direction of the field repeats is called the *wavelength*. For instance, points A and B in the figure are exactly one wavelength apart.

If you sit at a fixed point and watch the electric field as it passes by, you will again find that its direction oscillates. The time that it takes the pattern to repeat once is called the *period* of the wave, and is equal to the time that the source charge takes to repeat one cycle of its motion. The period is also equal to the time that the wave takes to travel a distance of one wavelength. Since it moves at the speed of light, we can infer that the wavelength and the period are related by

$$\text{speed} = \frac{\text{wavelength}}{\text{period}} \qquad \text{or} \qquad c = \frac{\lambda}{T}, \tag{5.1}$$

where λ ("lambda") is the standard symbol for wavelength, T is the standard symbol for period, and c is the speed of light. The *frequency* of an oscillation or a wave is the reciprocal of the period.

For reasons of tradition and convenience, electromagnetic waves of different wavelengths go by different names. *Radio waves*, with wavelengths of a meter or more, are generated relatively easily by running charge up and down an antenna. Somewhat shorter wavelengths are used for television and microwave communication. *Infrared* denotes wavelengths of a millimeter down to 700 nanometers; the random microscopic motions present in all matter at room temperature cause the emission of infrared radiation with wavelengths in the range of about a hundredth of a millimeter. Hotter objects, such as the sun, give off radiation in the *visible* spectrum, which covers the range 400–700 nanometers over which the human eye is sensitive. The wavelength of visible light determines its color, with red light having the longest wavelength and violet having the shortest (and the others following the order of the rainbow). Still shorter wavelengths are referred to successively as *ultraviolet*, *x-rays*, and *gamma rays*.

Exercise 5.1. Calculate the wavelength of the waves broadcast by an FM station whose frequency is 90.9 megahertz. (One megahertz is 10^6 s^{-1}.)

Question 5.2. Why is it most efficient for a radio transmission antenna to be oriented vertically?

5.2 Why is the Sky Blue?

The sun gives off visible light of all colors, which bombards the earth's atmosphere from above. The atmosphere is relatively transparent to most of this light. But if the atmosphere were completely transparent, the sky would appear black. Apparently, some of the light from the sun is scattered, or deflected, by air molecules. When we look at the sky in a direction away from the sun we see this scattered light, which is mostly blue (see figure 5.2). Conversely, red light is transmitted more easily by the atmosphere, making the sun appear red when it is near the horizon, since the light must pass through a great deal of air.

But *why* do air molecules scatter blue light more than red? Apparently, short wavelengths are scattered much more strongly than long wavelengths. We can understand this

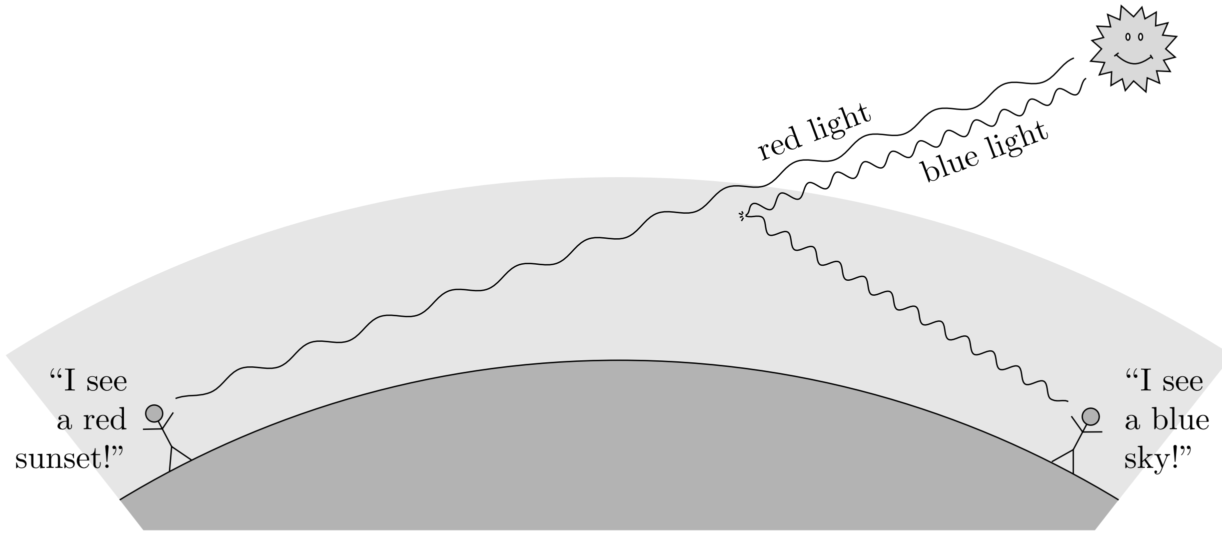

Figure 5.2. Air molecules scatter blue light more readily than red light, causing the sky to appear blue and the sun (especially when it is near the horizon) to appear red.

phenomenon by imagining a simple model of the scattering process, and applying the result (4.5) from the previous chapter that the energy radiated by an accelerated charge is proportional to the square of the acceleration.

Consider a single atom of nitrogen or oxygen in the atmosphere. For our purposes it is best to think of the atom as a tiny point of positive charge (the nucleus), surrounded by a larger cloud of smeared out negative charge (the electrons). The charges cancel, and the atom is electrically neutral. Now suppose an electromagnetic wave comes by. The electric field at the atom's location first points up, then down, then up again, down again, and so on. (For visible light, the wavelength is much larger than the size of an atom.) Although the neutral atom feels no net force from this electric field, its constituents do feel forces, so they are pulled slightly in opposite directions. They don't go far, however, since they pull back on each other. It is as if the electrons and the nucleus were attached together by a stiff spring.

As the wave passes by, the nucleus oscillates slightly up and down at the same frequency as the wave. We can describe its position as a function of time with a cosine function:

$$x(t) = x_0 \cos(\omega t), \tag{5.2}$$

where $\omega = 2\pi c/\lambda$ and λ is the wavelength. As long as the "spring" is very stiff, the amplitude x_0 will depend only on the strength of the electric field, not on the wavelength.

Since it is oscillating up and down, the nucleus itself gives off electromagnetic radiation, with the same frequency and wavelength. According to equation (4.5), the energy radiated is proportional to the square of the acceleration. The acceleration of the nucleus equals the second derivative of its position:

$$a(t) = \frac{d^2x}{dt^2} = -x_0\omega^2 \cos(\omega t). \tag{5.3}$$

We can now determine how the amount of energy radiated depends on the wavelength:

$$\text{Energy} \propto a^2 \propto \omega^4 \propto \frac{1}{\lambda^4}. \tag{5.4}$$

This formula says that a short-wavelength wave causes the nucleus to radiate much more energy than a long-wavelength wave. The same is true about the radiation given off by the electrons, which are oscillating in the opposite direction at the same frequency.

This electromagnetic radiation given off by the atom carries away energy, and the energy must come from somewhere. I think it should be plausible that the energy comes from the incoming wave that made the atom oscillate in the first place. As this wave continues on its way, some of its energy has been removed. I'd rather not go into the precise mechanism behind this process at this stage; if you believe in energy conservation, it has to happen somehow.

Thus we can conclude that when a light wave comes by, the atom removes some energy from it, and re-radiates this energy as a wave of the same wavelength, moving out in all directions. From equation (5.4) we see that this process is much more efficient for short wavelengths (i.e., violet and blue light) than for long wavelengths. This is why the sky is blue. Conversely, when a mixture of different colors of light passes through a lot of air, much of the blue light will be removed, leaving mostly red behind. This is why sunsets are red.

Question 5.3. Violet light has an even shorter wavelength than blue light. Why do you think the sky appears blue rather than violet? (I can think of three reasons.)

5.3 The Rutherford Atom

Exercise 5.4. In the Rutherford model of the hydrogen atom (popular early in this century), the electron orbits the proton in a circle of radius .53 Angstroms. (a) Use Coulomb's law and your knowledge of circular motion to find the speed of the electron in its orbit (answer: somewhat less than 1% of the speed of light). (b) Compute the kinetic energy of the electron in electron-Volts (answer: 13.6). (c) Compute the power radiated by the electron in electron-Volts per second. (d) If the electron loses energy continuously at this rate, how long does it take to lose all its kinetic energy? (Answer: .05 nanoseconds.) What happens to it after that? Comment on the plausibility of the Rutherford model.